



Homotopy perturbation transform algorithm for solving (2+1) dimensional type of the Zakharov–Kuznetsov equations

O. W. Lawal^{*a}, O. Z. Jagun^b, and L. M. Erinle-Ibrahim^a

^aDepartment of Mathematics, Tai Solarin University of Education Ijagun, Ogun State, Nigeria.

^bDepartment of Computer and Electrical Engineering, Olabisi Onabanjo University, Ago Iwoye, Ogun State, Nigeria.

Abstract

In this research, homotopy perturbation transform method (HPTM) is used to present the approximate analytical solutions of (2+1) dimensional type of the Zakharov–Kuznetsov nonlinear partial differential equations. This method gives solutions without any linearization and discretization or restrictive assumptions. Special cases of and are chosen as examples to show the capability and efficiency of the method. Maple 19.0 software is employed to compute the series generated from the algorithm. The results show that HPTM is very simple, reliable and effective in solving nonlinear problems.

Keywords: Zakharov–Kuznetsov equation; Homotopy perturbation method; Laplace transform method, Nonlinear differential equation; Algorithm; He's polynomial

1. Introduction

Zakharov-Kuznetsov equation of the form

$$u_t + \alpha(u^m)_x + \beta(u^n)_{xxx} + \gamma(u^k)_{yyx} = 0 \quad m, n, k \neq 0 \quad (1)$$

governs the behavior of weakly nonlinear ion-acoustic waves in plasma and it comprises cold ions and hot isothermal electrons in the presence of a uniform magnetic field [1, 2]. In the equation above, α, β, γ are arbitrary constants and m, n, k are integers. The ZK equation was first derived for describing weakly nonlinear ion-acoustic waves in strongly magnetized lossless plasma in two dimensions [3].

In the past year, the convergence of ZK equation by homotopy analysis method (HAM) was investigated with a proved theorem to show the convergence of HAM and the series solution of this equation via a reliable algorithm [4]. In [5], the approximate analytical solution of an ZK (m, n, k) equation was presented with the aid of the differential transform method (DTM). With this technique a solution was found in the form of a rapidly convergent series with easily computed components. Homotopy perturbation method (HPM) was applied to solve ZK(m, n, k) equations [6]. The result show that the method is very effective. This HPM was developed by combining the standard homotopy and perturbation method for solving linear and nonlinear equations without discretization, transformation or restrictive assumption [7]. Since Laplace transform cannot address nonlinear equations because of the problems caused by the nonlinear terms, several methods have been used recently to deal with these nonlinearities such as the Adomian decomposition method in [8] and the Laplace decomposition algorithm [9,10,11].

In this study, we combined the homotopy perturbation method with the He's polynomials (i.e semi analytical methods) and well-known Laplace transformation method (LTM) which is an analytical method as used in [13] to produce a highly efficient algorithm and technique for solving (2+1) Dimensional type of the Za-

kharov–Kuznetsov Equations. Homotopy perturbation transform method (HPTM) technique has the capability of providing the solution in a rapid convergent series which may lead to the solution in a closed form. The advantage of this method is its capability of combining two methods for obtaining good approximate solutions for solving (2+1) Dimensional type of the Zakharov–Kuznetsov Equations.

The advantage of HPM is that it does not require the initial and boundary conditions like some conservative methods rather it provides an exact solution by using only the initial conditions. The boundary conditions can be used only to justify the obtained result. Through tables, we will show that the method is equally able to arrive at approximate solutions of Zakharov–Kuznetsov Equations as obtained in [14].

Some basic concepts of HPTM were illustrated by consider a general nonlinear partial differential equation with the initial conditions [16].

2. Experimental evaluation

To illustrate its basic concepts of HPTM, we present the following examples.

2.1. Example 1

Consider the equation in the following form:

$$u_t + (u^2)_x + \frac{1}{8}(u^2)_{xxx} + \frac{1}{8}(u^2)_{yyx} = 0 \quad (2)$$

The exact solution of equation (2) subject to initial condition

$$u(x, y, 0) = \frac{4}{3}\lambda \sinh^2(x + y) \quad (3)$$

where λ is an arbitrary constant, was derived in [15] and is given as:

$$u(x, y, t) = \frac{4}{3}\lambda \sinh^2(x + y - \lambda t) \quad (4)$$

^{*}Corresponding author. Email: waheedlawal207@yahoo.com

$$L[u(x, y, t)] = \frac{u_0(x, y, 0)}{s} - \frac{L[(u^2)_x]}{s} - \frac{1}{8} \frac{L[(u^2)_{xxx}]}{s} - \frac{1}{8} \frac{L[(u^2)_{yyx}]}{s} \quad (5)$$

The inverse Laplace transform of equation (5) gives

$$u(x, y, t) = \frac{L^{-1}}{s} u_0(x, y, 0) - L^{-1} \left[\frac{L[(u^2)_x]}{s} \right] - \frac{1}{8} L^{-1} \left[\frac{L[(u^2)_{xxx}]}{s} \right] - \frac{1}{8} L^{-1} \left[\frac{L[(u^2)_{yyx}]}{s} \right] \quad (6)$$

Let's assume that the solution of equation (6) has the form

$$u(x, y, t) = \lim_{p \rightarrow 1} p^n u_n(x, y, t) = \sum_{n=0}^{\infty} p^n u_n \quad (7)$$

Now, applying the homotopy perturbation method to equation (6) and substituting equation (7) in to equation (6), this gives

$$\sum_{n=0}^{\infty} p^n u_n = \frac{L^{-1}}{s} u_0(x, y, 0) - p \left[L^{-1} \left[\frac{L[(\sum_{n=0}^{\infty} p^n u_n)^2]_x}{s} \right] + \frac{1}{8} L^{-1} \left[\frac{L[(\sum_{n=0}^{\infty} p^n u_n)^2]_{xxx}}{s} \right] + \frac{1}{8} L^{-1} \left[\frac{L[(\sum_{n=0}^{\infty} p^n u_n)^2]_{yyx}}{s} \right] \right] \quad (8)$$

By expanding equation (8) and comparing the coefficient of like of power of p , we obtain the following

$$p^0 : \frac{L^{-1}}{s} u_0(x, y, 0) \quad (9)$$

$$p^1 : -L^{-1} \left[\frac{L[(u_0)^2]_x}{s} \right] + \frac{1}{8} L^{-1} \left[\frac{L[(u_0)^2]_{xxx}}{s} \right] + \frac{1}{8} L^{-1} \left[\frac{L[(u_0)^2]_{yyx}}{s} \right] \quad (10)$$

$$p^2 : -2L^{-1} \left[\frac{L[(u_0 u_1)]_x}{s} \right] + \frac{1}{4} L^{-1} \left[\frac{L[(u_0 u_1)]_{xxx}}{s} \right] + \frac{1}{4} L^{-1} \left[\frac{L[(u_0 u_1)]_{yyx}}{s} \right] \quad (11)$$

$$p^3 : -L^{-1} \left[\frac{L[(u_1)^2]_x}{s} \right] - 2L^{-1} \left[\frac{L[(u_0 u_2)]_x}{s} \right] + \frac{1}{4} L^{-1} \left[\frac{L[(u_1)^2]_{xxx}}{s} \right] + \frac{1}{4} L^{-1} \left[\frac{L[(u_0 u_2)]_{xxx}}{s} \right] + \frac{1}{4} L^{-1} \left[\frac{L[(u_1)^2]_{yyx}}{s} \right] + \frac{1}{4} L^{-1} \left[\frac{L[(u_0 u_1)^2]_{yyx}}{s} \right] \quad (12)$$

Solving the system of equation (9) – (12) using Maple 19.0, we obtain the following

$$u_0 := \frac{4}{3} \lambda \sinh^2(x + y) \quad (13)$$

$$u_1 := \frac{32}{9} \lambda^2 t \sinh(2y + 2x) - \frac{49}{9} t \lambda^2 \sinh(4x + 4y) \quad (14)$$

$$u_2 := \frac{272}{9} \lambda^3 t^2 \cosh(2y + 2x) - \frac{608}{9} \lambda^3 t^2 \cosh(4x + 4y) + \frac{1040}{27} \lambda^3 t^2 \cosh(6x + 6y) - \frac{32}{27} \lambda^3 t^2 \quad (15)$$

$$u_3 := -\frac{32768}{81} \lambda^4 t^3 \sinh(6y + 6x) - \frac{22720}{243} \lambda^4 t^3 \sinh(4x + 4y) + \frac{16384}{243} \lambda^4 t^3 \sinh(2x + 2y) - \frac{192160}{243} \lambda^4 t^2 \sinh(8y + 8x) \quad (16)$$

and so on for other components. The solution after the fourth iteration as is giving by

$$u(x, y, t) = \sum_{n=0}^3 p^n u_n = u_0 := \frac{4}{3} \lambda \sinh^2(x + y) + \frac{32}{9} \lambda^2 t \sinh(2y + 2x) - \frac{49}{9} t \lambda^2 \sinh(4x + 4y) + \frac{272}{9} \lambda^3 t^2 \cosh(2y + 2x) - \frac{608}{9} \lambda^3 t^2 \cosh(4x + 4y) + \frac{1040}{27} \lambda^3 t^2 \cosh(6x + 6y) - \frac{32}{27} \lambda^3 t^2 - \frac{32768}{81} \lambda^4 t^3 \sinh(6y + 6x) - \frac{22720}{243} \lambda^4 t^3 \sinh(4x + 4y) + \frac{16384}{243} \lambda^4 t^3 \sinh(2x + 2y) - \frac{192160}{243} \lambda^4 t^2 \sinh(8y + 8x) \quad (17)$$

2.2. Example 2

Consider the ZK (3,3,3) equation in the following form:

$$u_t + (u^3)_x + 2(u^3)_{xxx} + 2(u^3)_{yyx} = 0, \tag{18}$$

The exact solution of equation (18) subject to initial condition

$$u(x, y, 0) = \frac{3}{2} \lambda \sinh \left[\frac{1}{6} (x + y) \right] \tag{19}$$

where λ is an arbitrary constant, was derived in [15] and is given as:

$$u(x, y, t) = \frac{3}{2} \lambda \sinh \left[\frac{1}{6} (x + y - \lambda t) \right] \tag{20}$$

To apply HPTM, we take the Laplace transform of equation (18) subject to the initial condition (19), we have

$$L[u(x, y, t)] = \frac{u_0(x, y, 0)}{s} - \frac{L[(u^3)_x]}{s} - 2 \frac{L[(u^3)_{xxx}]}{s} - 2 \frac{L[(u^3)_{yyx}]}{s} \tag{21}$$

$$\sum_{n=0}^{\infty} p^n u_n = \frac{L^{-1}}{s} u_0(x, y, 0) - p \left[L^{-1} \left[\frac{L[(\sum_{n=0}^{\infty} p^n u_n)^2]_x}{s} \right] + 2L^{-1} \left[\frac{L[(\sum_{n=0}^{\infty} p^n u_n)^2]_{xxx}}{s} \right] + 2L^{-1} \left[\frac{L[(\sum_{n=0}^{\infty} p^n u_n)^2]_{yyx}}{s} \right] \right] \tag{24}$$

By expanding equation (24) and comparing the coefficient of like of power of p , we obtain the following

$$p^0 : \frac{L^{-1}}{s} u_0(x, y, 0) \tag{25}$$

$$p^1 : -L^{-1} \left[\frac{L[(u_0^3)_x]}{s} \right] + 2L^{-1} \left[\frac{L[(u_0^3)_{xxx}]}{s} \right] + 2L^{-1} \left[\frac{L[(u_0^3)_{yyx}]}{s} \right] \tag{26}$$

$$p^2 : -3L^{-1} \left[\frac{L[(u_0^2 u_1)_x]}{s} \right] + 6L^{-1} \left[\frac{L[(u_0^2 u_1)_{xxx}]}{s} \right] + 6L^{-1} \left[\frac{L[(u_0^2 u_1)_{yyx}]}{s} \right] \tag{27}$$

$$p^3 : -3L^{-1} \left[\frac{L[(u_0^2 u_2)_x]}{s} \right] - 3L^{-1} \left[\frac{L[(u_0 u_1^2)_x]}{s} \right] + 6L^{-1} \left[\frac{L[(u_0 u_1^2)_{xxx}]}{s} \right] + 6L^{-1} \left[\frac{L[(u_0^2 u_2)_{xxx}]}{s} \right] + 6L^{-1} \left[\frac{L[(u_0^2 u_2)_{yyx}]}{s} \right] + 6L^{-1} \left[\frac{L[(u_0 u_1^2)_{yyx}]}{s} \right] \tag{28}$$

Solving the system of equation (25) - (28) using Maple 19.0, we obtain the following

$$u_0 := \frac{3}{2} \lambda \sinh \left(\frac{1}{6} x + \frac{1}{6} y \right) \tag{29}$$

$$u_1 := \frac{3}{8} \lambda^3 t \cosh \left(\frac{1}{6} x + \frac{1}{6} y \right) \tag{30}$$

$$u_2 := \frac{45}{512} \lambda^5 t^2 \sinh \left(\frac{1}{6} x + \frac{1}{6} y \right) - \frac{51}{512} \lambda^5 t^2 \sinh \left(\frac{1}{2} x + \frac{1}{2} y \right) \tag{31}$$

$$u_3 := -\frac{51}{4096} \lambda^7 t^3 \cosh \left(\frac{1}{6} x + \frac{1}{6} y \right) - \frac{65}{2048} \lambda^7 t^3 \cosh \left(\frac{1}{2} x + \frac{1}{2} y \right) + \frac{17}{4096} \lambda^7 t^3 \cosh \left(\frac{5}{6} x + \frac{5}{6} y \right) \tag{32}$$

and so on for other components. The solution after the fourth iteration as $p \rightarrow 1$ is giving by

$$u(x, y, t) = \sum_{n=0}^3 p^n u_n = \frac{3}{2} \lambda \sinh \left(\frac{1}{6} x + \frac{1}{6} y \right) + \frac{3}{8} \lambda^3 t \cosh \left(\frac{1}{6} x + \frac{1}{6} y \right) + \frac{45}{512} \lambda^5 t^2 \sinh \left(\frac{1}{6} x + \frac{1}{6} y \right) - \frac{51}{512} \lambda^5 t^2 \sinh \left(\frac{1}{2} x + \frac{1}{2} y \right) - \frac{51}{4096} \lambda^7 t^3 \cosh \left(\frac{1}{6} x + \frac{1}{6} y \right) - \frac{65}{2048} \lambda^7 t^3 \cosh \left(\frac{1}{2} x + \frac{1}{2} y \right) + \frac{17}{4096} \lambda^7 t^3 \cosh \left(\frac{5}{6} x + \frac{5}{6} y \right) \tag{33}$$

The inverse Laplace transform of equation (21) gives

$$u(x, y, t) = \frac{L^{-1}}{s} u_0(x, y, 0) - L^{-1} \left[\frac{L[(u^3)_x]}{s} \right] - 2L^{-1} \left[\frac{L[(u^3)_{xxx}]}{s} \right] - 2L^{-1} \left[\frac{L[(u^3)_{yyx}]}{s} \right] \tag{22}$$

Let's assume that the solution of equation (22) has the form

$$u(x, y, t) = \lim_{p \rightarrow 1} p^n u_n(x, y, t) = \sum_{n=0}^{\infty} p^n u_n \tag{23}$$

Now, applying the homotopy perturbation method to equation (22) and substituting equation (23) in to equation (22), this gives

Table 1: Solutions using the 4-iteration of HPTM.

x	y	t	Example 1			Example 2		
			Present	From [14]	Exact	Present	From [14]	Exact
0.1	0.1	0.2	5.355135576E-5	5.355355975E-5	5.393877159E-5	5.000933482E-5	5.000914112E-5	4.995923204E-5
		0.3	5.330325430E-5	5.330816448E-5	5.388407669E-5	5.000937234E-5	5.000915456E-5	4.993421817E-5
		0.4	5.305539226E-5	5.306406852E-5	5.382941057E-5	5.000940986E-5	5.000910586E-5	4.990920434E-5
0.6	0.6	0.2	2.988769738E-3	2.989873669E-3	3.036507411E-3	3.020040803E-4	3.020039162E-4	3.019530008E-4
		0.3	2.964911397E-3	2.967173317E-3	3.035778955E-3	3.020041186E-4	3.020038551E-4	3.019274992E-4
		0.4	2.941590341E-3	2.945226366E-3	3.035050641E-3	3.020041568E-4	3.020037937E-4	3.019019978E-4
0.9	0.9	0.2	1.100112510E-2	1.102484681E-2	1.153697757E-2	4.567805185E-4	4.567802934E-4	4.567281735E-4
		0.3	1.076823444E-2	1.079635470E-2	1.153454074E-2	4.567805577E-4	4.567802556E-4	4.567020404E-4
		0.4	1.056592099E-2	1.057416210E-2	1.153210438E-2	4.567805969E-4	4.567801785E-4	4.566759074E-4

3. Results and discussion

An approximate solution of equation (2) and (18) subject to initial condition (3) and (19) respectively are presented using HPTM. Also, to show the accuracy and convergence of HPTM, the exact solution, the variation iteration method (VIM) solution obtained in [14] and the present solution are illustrated in Table 1. It is observed that the accuracy of HPTM for $ZK(m, n, k)$ equation is controllable and its results are more closer to exact solution depending on the choice of x, y and t . Therefore, if more terms of HPTM are computed, it is possible that the numerical results obtained will correspond to exact solutions.

4. Conclusion

In this research, we have successfully developed an algorithm for solving (2+1) Dimensional type of the Zakharov–Kuznetsov Equations with specific initial conditions using the HPTM. The solution obtained after 4th iteration shows that it will be closer to exact solution if more terms can be computed. Therefore, HPTM has ability of reducing the volume of the computational work considerably compared to other methods while still maintaining the high accuracy of the numerical result. Because of its reliability and applicability, the method can be considered as a good refinement of existing numerical techniques and can be applied to solve some nonlinear differential equations.

References

- [1] Monro S & Parkes E J, The derivation of a modified Zakharov Kuznetsov equation and the stability of its solutions, *Journal of Plasma Physics*, 62(3) (1999) 305–317.
- [2] Monro S & Parkes E J, Stability of solitary-wave solutions to a modified ZakharovKuznetsov equation, *Journal of Plasma Physics*, 64 (3) (2000) 411–426.
- [3] Zakharov V E & Kuznetsov E A, Three-dimensional solitons, *Soviet Physics JETP*, 39 (1974) 285–286.
- [4] Fallahzadeha A & Fariborzi M A, A note on the convergence of the Zakharov-Kuznetsov equation by homotopy analysis method, *Journal of Linear and Topological Algebra*, 3(1) (2014) 7–13.
- [5] Hesam S, Nazemi A & Haghbin A, Analytical solution for the Zakharov-Kuznetsov equations by differential transform method, *International Science Index, Mathematical and Computational Sciences*, 5(3) (2011).
- [6] Biazar J, Badpeima F. & Azimi F, Application of the homotopy perturbation method to Zakharov-Kuznetsov equations, *Computer and Mathematics with Applications*, 58 (2009) 2391–2394.
- [7] He J H, Homotopy perturbation method: a new nonlinear analytical technique, *Applied Mathematics and Computation*, 135 (2003) 73–79.
- [8] Biazar J, Gholami M & Ghanbari B, Extracting a general iterative method from an Adomian decomposition method and comparing it to the variational iteration method, *Computers & Mathematics with Applications*, 59 (2010) 622–628.
- [9] Khuri S A, A Laplace decomposition algorithm applied to a class of nonlinear differential equations, *Journal of Applied Mathematics*, 1 (2010) 141–155.
- [10] Yusufoglu E, Numerical solution of Duffing equation by the Laplace decomposition algorithm, *Applied Mathematics and Computation*, 177 (2006) 572–580.
- [11] Islam S, Faraz Y K & Austin F, Numerical solution of logistic differential equations by using the Laplace decomposition method, *World Applied Sciences Journal*, 8 (2010) 1100–1105.
- [12] Madani M & Fathizadeh M, Homotopy perturbation algorithm using Laplace transformation, *Nonlinear Science Letters A*, 1 (2010) 263–267.
- [13] Molliq A Y, Noorani M S, Hashim I & Ahmad R R, Approximate solutions of fractional Zakharov–Kuznetsov equations by VIM, *Journal of Computational and Applied Mathematics*, 233 (2009) 103–108.
- [14] Inc M, Exact solutions with solitary patterns for the Zakharov–Kuznetsov equations with fully nonlinear dispersion, *Chaos Solitons Fractals*, 33(5) (2007) 1783–1790.
- [15] Khan Y & Wu Q, Homotopy perturbation transform method for nonlinear equations using He’s polynomials, *Computer and Mathematics with Applications*, 61(8) (2011) 1963–1967.