



The value of the customer's waiting time for finite source queueing system

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Abstract

In finite source queueing system, the value of customer's waiting time is crucial because it impacts the system's efficiency, service quality, and overall performance. Past studies used a derivative free Powell's method for optimization of finite queue model, but these studies do not explicitly focus on optimization of the average numbers of customers in the waiting line as a function of traffic intensity. This paper presents the optimization mathematical model to calculate the value of customer's waiting time for finite source queueing system by the help of calculus. To develop this model, we have made some assumptions in service discipline, service time, and model system.

Keywords: Finite source queue; Optimization; Waiting time.

1. Introduction

Fixed number of customers are served in finite source queueing system where arrival rate, waiting time, and traffic intensity are given more emphasis. Finite source queueing system is widely applicable for fixed customer's service centre such as those in health care, manufacturing and repairing, public service, aviation and transportation. In such a case, ignoring the value of customer's waiting time can have a serious negative impact on the goodwill and trust, policy and service system, service quality, resource and capacity planning, overall efficiency, and satisfaction of customers which are highly valuable for the growth of service centres and the long term operating strategy of service system. Thus, finding the value of the customer's waiting time is important for finite source queue system.

Past studies used a derivative free Powell's method for optimization of finite queue model, but these studies do not explicitly focus on optimization of the average number of customers in the waiting line as a function of traffic intensity. For example, Cruz and Woensed [1] explored the wide application of Powell's method on finite queueing modelling and optimization. Derivative based optimization, especially in the case of sensitivity analysis, is highly accurate, smooth, and fast. Nevertheless, Singhal et. al [2] proposed a framework for measuring customer waiting time valuation, but their study seems to focus on infinite source queueing systems. In addition, [3] argue on "enhancing finite-difference based derivative-free optimization methods with machine learning" but it is still unclear on the case of sensitivity analysis. This indicates that there is a research gap regarding customer waiting time valuation within finite source queueing system.

To represent such a system practically, the $M/M/s/C/N$ model is used, where the arrival process is Poisson, the service time follows an exponential distribution, the system has s finite servers, only a maximum of C customers can be in the system at a time, and the total possible number of customers is limited by N [4]. In this

context, this study presents a new mathematical approach to find the value of the customer's waiting time for the $M/M/s/C/N$ finite source queueing model with respect to server utilization factor using a calculus based optimization framework.

As a model, the flow structure of a finite source queueing system shown in Figure 1 clearly shows the interrelationship between arrival, waiting, service, and blocking, where customers with a finite population (N) enter the system according to the arrival rate (λ), wait in the waiting area when the server is busy, and blocking and opportunity loss occur when the system reaches full.

In this study the model $M/M/s/C/N$ is capable to address the limited server capacity, limited waiting spaces or waiting buffer ($C - s$) and congestion which are really found in real service system. Since the waiting phenomenon does not occur in traditional models such as $M/G/\infty/N/N$ with infinite servers, the evaluation of waiting time is not possible in such models [5]. Therefore, this study presented within the $M/M/s/C/N$ framework that simultaneously encompasses finite servers, finite capacity, and finite population seems structurally different, theoretically sound, and more practically relevant. In this sense, the present research can be claimed to be unique as an analytical contribution to the evaluation of customer waiting time within finite source queueing theory. So, this paper presents the optimization mathematical model to calculate the value of customer's waiting time for finite source queueing system especially $M/M/s/C/N$ model with respect to traffic intensity or server's utilization factor.

2. The $M/M/s/C/N$ model and waiting line

Originally Engset[6] analysed for $s = 1$ in the $M/M/s$ model where the first M represents the Poisson arrival from finite source, second M represents the Markovian exponentially distributed service time, and s is the number of servers. Later on, Takagi [4] used the $M/M/s/C/N$ model in finite queueing analysis in which C represents system capacity and N represents the maximum number of potential customers. This model is also called finite source

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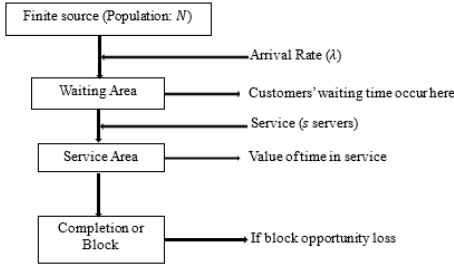


Figure 1: Finite queueing source model.

Note. The figure is developed from the theoretical perspective of finite queueing source system.

multi server Markovian queue.

Raicu et. al. [7] investigate $M/M/n:(m/FIFO)$ systems, essentially $M/M/s/C/N$ models combining waiting space and rejection, and provide “recommendations emphasise the necessity for further research on the mathematical modelling of finite capacity queueing systems” (p. 12). Here, the waiting line is formed in $M/M/s/C/N$ model when $n > s$. In addition, waiting customers are $(n - s)$ when servers are fully occupied. n denotes the total number of customers in the system at any given time, which includes both customers receiving service and those in the waiting area.

The value of n is always limited to the range $0 \leq n \leq C$. In such a case, while arrival rate is increased then the queue is also increased. That is traffic intensity is increased if queue length is also increased and the function of queue length depends on the traffic load, symbolically represented by $L_q(\rho)$. In the finite source queue, Gross et. al. (2008) [5] (p.86) express the “function of queue length L_q ” under the M finite population size and c servers as:

$$L_q = \sum_{n=c+1}^M (n - c) p_n.$$

As shown in equation 1, in the given expression, the queue length L_q and the steady state probability P_n are taken with respect to traffic intensity (ρ) having exactly n customers in system with s number of servers really shows the existence of queue when the summation starts from $n = s + 1$ to system capacity C where $C \leq N$ and we get:

$$L_q(\rho) = \sum_{n=s+1}^C (n - s) p_n(\rho) \quad (1)$$

Again, Gross et. al. [5] (p. 85) provide the following expression for “steady state probability in finite source queueing system under the M finite population size and c servers” as:

$$P_n = \begin{cases} \binom{M}{n} r^n \cdot P_0; & 1 \leq n \leq c, \\ \binom{M}{n} \frac{n!}{c^{n-c} \cdot c!} r^n \cdot P_0; & c \leq n \leq M. \end{cases}$$

Rearranging the given expression under the $M/M/s/C/N$ system by taking the case of $s \leq n \leq N$ with traffic intensity (ρ), and substituting ‘ c ’ by ‘ s ’ customers in the system and ‘ M ’ by ‘ N ’ finite population size, we get:

$$\beta = \frac{\left[\sum_{n=0}^{s-1} \binom{N}{n} \frac{(s-n) n! \rho^n}{s^{n-s} s!} \right] \left[\sum_{n=0}^{s-1} \binom{N}{n} \rho^n + \sum_{n=s}^N \binom{N}{n} \frac{n!}{s^{n-s} s!} \rho^n \right] - \left[\sum_{n=0}^{s-1} \binom{N}{n} \frac{(s-n) n! \rho^n}{s^{n-s} s!} \right] \left[\sum_{n=0}^{s-1} \binom{N}{n} n \rho^{n-1} + \sum_{n=s}^N \binom{N}{n} \frac{n! n \rho^{n-1}}{s^{n-s} s!} \right]}{\left[\sum_{n=s+1}^C \binom{N}{n} \frac{(n-s) n! \rho^n \cdot n}{s^{n-s} s!} \right] \left[\sum_{n=0}^{s-1} \binom{N}{n} \rho^n + \sum_{n=s}^N \binom{N}{n} \frac{n!}{s^{n-s} s!} \rho^n \right] - \left[\sum_{n=s+1}^C \binom{N}{n} \frac{(n-s) n! \rho^n}{s^{n-s} s!} \right] \left[\sum_{n=0}^{s-1} \binom{N}{n} n \rho^{n-1} + \sum_{n=s}^N \binom{N}{n} \frac{n! n \rho^{n-1}}{s^{n-s} s!} \right]}$$

$$P_n(\rho) = \binom{N}{n} \frac{n!}{s^{n-s} s!} \cdot \rho^n \cdot P_0; \quad s \leq n \leq N \quad (2)$$

Where $p_n(\rho)$ denotes the steady state probability of having exactly ‘ n ’ customers in the system at traffic intensity (ρ).

The concept $\sum_{n=0}^N P_n = 1$ is plugged into the Gross et al., [5] steady state probability under both conditions $1 \leq n \leq s$ and $s \leq n \leq N$. Then we get the value of P_0 :

$$P_0 \sum_{n=0}^{s-1} \binom{N}{n} \rho^n + P_0 \sum_{n=s}^N \binom{N}{n} \frac{n!}{s^{n-s} s!} \rho^n = 1$$

$$\therefore P_0 = \frac{1}{\left[\sum_{n=0}^{s-1} \binom{N}{n} \rho^n + \sum_{n=s}^N \binom{N}{n} \frac{n!}{s^{n-s} s!} \rho^n \right]} \quad (3)$$

3. The value of the customer’s waiting time (β)

The value of the customer’s waiting time plays an important role in the system’s performance, and each queue holds the value of customer’s waiting time. As Singhal et al. [2] state that the value of the customer’s waiting time is the ratio of marginal changes in the mean server idle time (S_s) and marginal changes in the average length of the queue of waiting customers (L_q), which is different than the cost of the customer’s waiting time. Further, the mean server idle time (S_s) is also calculated under $M/M/s/C/N$ finite source queue system by taking the concept of Ross [8] (p.~9): “expectation of the random variable X , denoted by $E[X]$, is defined by $\sum_x x p\{X = x\}$ ”. Where $p\{X = x\}$ is steady state probability. Then we get equation 4 as,

$$S_s(\rho) = \sum_{n=0}^{s-1} (s - n) P_n(\rho) \quad (4)$$

Where $S_s(\rho)$ denotes the average number of idle servers in the system at traffic intensity ρ . When the number of customers in the system n is less than the number of servers s , then $(s - n)$ servers remain idle. In equation 4, the summation starts from $n = 0$ to $s - 1$, because only within this range it is possible for a server to become idle. When the number of customers in the system n is equal or greater than the number of servers s , then all servers are busy and the number of idle servers is zero. In addition, $p_n(\rho)$ in equations 1 and 4 denote the same steady state probability of having exactly n customers in the system under traffic intensity ρ .

Singhal et al., [2] (p. 570) provide the following expression for the value of customer’s waiting time:

$$\beta = - \frac{dS_s}{dL_q} = - \frac{\frac{d(S_s)}{d\rho}}{\frac{d(L_q)}{d\rho}}$$

The derivative of S_s and L_q with respect to the traffic intensity under $M/M/s/C/N$ finite queue system are calculated and substituted into the given expression β , and it was found that the final expression as theorem of the value of customer’s waiting time for $M/M/s/C/N$ finite model is:

Theorem 1. The value of customer’s waiting time in finite queue is,

Proof. In equations 1 and 4, substitute the value of $P_n(\rho)$ from equation 2 with the value of P_0 from equation 3 respectively. Then we get:

$$L_q(\rho) = \sum_{n=s+1}^C (n-s) \cdot \binom{N}{n} \frac{n!}{s^{n-s} s!} \rho^n \cdot \frac{1}{\left[\sum_{n=0}^{s-1} \binom{N}{n} \rho^n + \sum_{n=s}^N \binom{N}{n} \frac{n!}{s^{n-s} s!} \rho^n \right]} \tag{5}$$

$$S_s(\rho) = \sum_{n=0}^{s-1} (s-n) \cdot \binom{N}{n} \frac{n!}{s^{n-s} s!} \rho^n \cdot \frac{1}{\left[\sum_{n=0}^{s-1} \binom{N}{n} \rho^n + \sum_{n=s}^N \binom{N}{n} \frac{n!}{s^{n-s} s!} \rho^n \right]} \tag{6}$$

Differentiate both sides of equation 5 with respect to ρ . Then we get:

$$\frac{d[L_q(\rho)]}{d\rho} = \frac{d \left\{ \frac{\sum_{n=s+1}^C (n-s) \binom{N}{n} \frac{n!}{s^{n-s} s!} \rho^n}{\left[\sum_{n=0}^{s-1} \binom{N}{n} \rho^n + \sum_{n=s}^N \binom{N}{n} \frac{n!}{s^{n-s} s!} \rho^n \right]} \right\}}{d\rho} \tag{7}$$

Applying the quotient rule to equation 7:

$$\frac{d[L_q(\rho)]}{d\rho} = \frac{\left[\sum_{n=s+1}^C \binom{N}{n} \frac{(n-s) n! \rho^n \cdot n}{s^{n-s} s! \rho} \right] \left[\sum_{n=0}^{s-1} \binom{N}{n} \rho^n + \sum_{n=s}^N \binom{N}{n} \frac{n!}{s^{n-s} s!} \rho^n \right] - \left[\sum_{n=s+1}^C \binom{N}{n} \frac{(n-s) n! \rho^n}{s^{n-s} s!} \right] \left[\sum_{n=0}^{s-1} \binom{N}{n} n \rho^{n-1} + \sum_{n=s}^N \binom{N}{n} \frac{n! n \rho^{n-1}}{s^{n-s} s!} \right]}{\left[\sum_{n=0}^{s-1} \binom{N}{n} \rho^n + \sum_{n=s}^N \binom{N}{n} \frac{n!}{s^{n-s} s!} \rho^n \right]^2} \tag{8}$$

Differentiate both sides of equation 6 with respect to ρ . Then we get:

$$\frac{d[S_s(\rho)]}{d\rho} = \frac{d \left\{ \frac{\sum_{n=0}^{s-1} (s-n) \binom{N}{n} \frac{n!}{s^{n-s} s!} \rho^n}{\sum_{n=0}^{s-1} \binom{N}{n} \rho^n + \sum_{n=s}^N \binom{N}{n} \frac{n!}{s^{n-s} s!} \rho^n} \right\}}{d\rho} \tag{9}$$

Applying the quotient rule to equation 9. Then we get:

$$\frac{d[S_s(\rho)]}{d\rho} = \frac{\left[\sum_{n=0}^{s-1} \binom{N}{n} \frac{(s-n) n! \rho^n \cdot n}{s^{n-s} s! \rho} \right] \left[\sum_{n=0}^{s-1} \binom{N}{n} \rho^n + \sum_{n=s}^N \binom{N}{n} \frac{n!}{s^{n-s} s!} \rho^n \right] - \left[\sum_{n=0}^{s-1} \binom{N}{n} \frac{(s-n) n! \rho^n}{s^{n-s} s!} \right] \left[\sum_{n=0}^{s-1} \binom{N}{n} n \rho^{n-1} + \sum_{n=s}^N \binom{N}{n} \frac{n! n \rho^{n-1}}{s^{n-s} s!} \right]}{\left[\sum_{n=0}^{s-1} \binom{N}{n} \rho^n + \sum_{n=s}^N \binom{N}{n} \frac{n!}{s^{n-s} s!} \rho^n \right]^2} \tag{10}$$

Substituting the values of equation 8 and equation 10 into $\beta = -\frac{d(S_s)/d\rho}{d(L_q)/d\rho}$.

Then we get:

$$\beta = -\frac{\left[\sum_{n=0}^{s-1} \binom{N}{n} \frac{(s-n) n! \rho^n \cdot n}{s^{n-s} s! \rho} \right] \left[\sum_{n=0}^{s-1} \binom{N}{n} \rho^n + \sum_{n=s}^N \binom{N}{n} \frac{n!}{s^{n-s} s!} \rho^n \right] - \left[\sum_{n=0}^{s-1} \binom{N}{n} \frac{(s-n) n! \rho^n}{s^{n-s} s!} \right] \left[\sum_{n=0}^{s-1} \binom{N}{n} n \rho^{n-1} + \sum_{n=s}^N \binom{N}{n} \frac{n! n \rho^{n-1}}{s^{n-s} s!} \right]}{\left[\sum_{n=s+1}^C \binom{N}{n} \frac{(n-s) n! \rho^n \cdot n}{s^{n-s} s! \rho} \right] \left[\sum_{n=0}^{s-1} \binom{N}{n} \rho^n + \sum_{n=s}^N \binom{N}{n} \frac{n!}{s^{n-s} s!} \rho^n \right] - \left[\sum_{n=s+1}^C \binom{N}{n} \frac{(n-s) n! \rho^n}{s^{n-s} s!} \right] \left[\sum_{n=0}^{s-1} \binom{N}{n} n \rho^{n-1} + \sum_{n=s}^N \binom{N}{n} \frac{n! n \rho^{n-1}}{s^{n-s} s!} \right]}$$

Proved

□

4. Conclusion and discussion

The aim of this paper is to present a novel model to calculate the value of customers' waiting time in a finite source queueing system, specifically the $M/M/s/C/N$ model, and it has fulfilled its aim. The developed model is particularly useful for finite source queueing systems and may be compatible with high monetary or service value. This is just a model and needs to be tested further for validation and for more authenticity. This model is also useful to address traffic management where the case of a finite population occurs. Similarly, we can use this model in hospitals, machine repair centers, and other places where the condition of this model is applicable. Moreover, it supports sensitivity analysis with respect to traffic intensity and arrival rate in an area identified as a gap in previous studies. Past studies like Singhal et al. [2] present the concept of finding the value of customer's waiting time but it was limited on infinite queueing model. This paper filled the gap.

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