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Exploring rheological variability in different geophysical flows

Shyamshwar Baral[a,b], Narayan Sapkota[a,c], Santosh Paudel[a], Parameshwari Kattel[d], and Khim B. Khattri*[a]

[a]Department of Mathematics, School of Science, Kathmandu University, Dhulikhel, Kavre, Nepal.
[b]Department of Mathematics, School of Science and Management, Gurukul Secondary School, Chitwan, Nepal.
[c]Department of Computer Engineering, Everest Engineering College, Sanepa, Lalitpur, Nepal.
[d]Tribhuvan University, Institute of Science and Technology, Tri-Chandra M. Campus, Kathmandu, Nepal.

Abstract

This study focuses on understanding how natural mass movements; such as debris flows, mudflows, and granular density currents; behave under varying environmental conditions. Rheology, the science of deformation and flow of matter, plays a central role in predicting flow dynamics. We analyse rheological properties across both Newtonian and non-Newtonian flow regimes. Within the non-Newtonian flow regimes, rheological models such as Bingham, Herschel-Bulkley, quadratic, and bilinear are studied, focusing the relationship between shear stress and shear rate across several parametric ranges. Furthermore, this paper primarily emphasizes a study of the theoretical foundations of several rheological models, including Bagnold's grain-inertia theory, Mohr–Coulomb plasticity, and the $\mu(I)$-rheology. Our comparative analyses highlight the critical need for adaptive, multi-phase rheological formulations in geophysical mass flow modelling. The insights gained contribute to bridging the gap between idealized rheology and field-scale flow behaviour, paving the way for next-generation models that incorporate dynamic solid-fluid interactions for more accurate hazard prediction and mitigation.

Keywords: Effective viscosity; Yield-stress; Viscoplasticity; Strain-rate; Shear-stress.

1. Introduction

Geophysical mass movements, including debris flows, landslides, avalanches, and pyroclastic surges, represent significant hazards to human populations and infrastructure, especially within mountainous areas characterized by steep slopes, intricate geological settings, and heavy rainfall [1, 2]. In Nepal, spanning from the towering Himalayas to the low-lying Terai plains, geophysical mass flows are commonly initiated by intense monsoonal rainfall, anthropogenic development work, and seismic disturbances, rendering hazard assessment and mitigation a critical national concern [3, 4]. Among these, debris flows pose severe threats due to their sudden onset, substantial volume, and intricate flow dynamics. Such flows consist of concentrated mixtures of solid particles and interstitial fluids, exhibiting non-Newtonian rheological behaviour in which viscosity varies with the applied stress or strain rate [5, 6, 7]. Depending on stress conditions, the granular materials involved can deform elastically, plastically, or viscously [8, 9, 10]. Accurately modelling this behaviour is inevitable for simulating flow mobility and path, impact pressure and forces, and deposition patterns along with inundation area. A variety of rheological models have been developed to capture the diverse mechanical responses of debris flows. Classical models such as Bingham [11, 12] and Herschel–Bulkley [13] account for yield stress and shear-dependent viscosity [14, 15, 16], while the Voellmy-Salm model [17, 18] incorporates frictional and turbulent resistance [19, 20]. While originally applied to soft matter systems such as food products [21], and biological fluids [22], rheological theory has become central to modeling geophysical mass flows. Granular flows such as avalanches are often treated as non-Newtonian fluids [1, 2], requiring non-linear stress–strain relationships. Another advance model is $\mu(I)$-rheology, which relates the internal friction coefficient to a dimensionless inertial number [23, 24, 25]. It is widely adopted because it is computationally tractable and has been successfully integrated into depth-averaged and full 3D simulation models, improving predictive capability while maintaining physical consistency. Moreover, it unifies granular flow behaviour across different regimes, from slow, quasi-static deformation to rapid, collisional flow within a single, experimentally validated constitutive law [23, 24, 26].

Despite these advances, most models depended on fixed rheological parameters (e.g., dynamic viscosity, yield stress, plastic viscosity, flow index, and friction angle) and struggle to represent the spatial and temporal variability seen in natural debris flows [9, 27]. Evolving factors such as pore pressure, fluid content, and solid concentration significantly influence flow behaviour [23, 27] and are not addressed by many classical models. In addition to this, theses models often fail to capture the transitions between flow regimes, making predictive simulations unreliable [27]. This has led to the development of more adaptive formulations, including pressure and rate-dependent viscoplastic models and mixture theories that integrate solid–fluid interactions [1, 28, 29]. Mixture theory, introduced by Iverson [27], conceptualizes debris flows as two-phase systems with distinct solid and fluid components. Based on this, subsequent models have incorporated multidimensional and multiphase mechanics to simulate complex flow behaviour [1, 30]. These can be broadly classified into (1) mixture models, which track interfacial dynamics between phases [10, 31], and (2) rheological models, which treat the material as a single-phase medium with non-linear constitutive laws [32, 33, 34]. While one-phase models are suitable for snow or rock avalanche or muddy flows

*Corresponding author. Email: khimkhattri@ku.edu.np

with high fluid content, two-phase models better represent flows with significant granular concentration [35, 36, 37]. The choice of rheological model significantly affects predictions of flow dynamics, run-out distance, and deposit geometry. Factors such as shear rate, local agitation, pore pressure, and interphase drag must be considered in selecting appropriate formulations [1, 15, 38, 39].

In the field of fluid mechanics and environmental engineering, the complexity of natural mass flows such as debris flows, lahars, and sediment-laden floods underscores the necessity for rigorous mathematical investigation. A systematic comparison and evaluation of rheological models under controlled conditions is essential to capture the interplay between fluid and solid phases, pore pressure dynamics, and interfacial stresses. This approach was pioneered by the GDR MiDi collaboration [26], which led to the development of unified rheological frameworks for dense granular flows [40, 41]. Mixture models have been widely employed to track inter-facial dynamics between phases [10, 31], whereas rheological models treat the material as a single-phase continuum governed by non-linear constitutive laws [9, 32, 34]. Furthermore, factors such as shear rate, local agitation, pore pressure, and interphase drag must be incorporated into model selection to ensure physically consistent and environmentally relevant simulations [38, 39].

In this study, we simulate the stress–strain behaviour of debris flows using a suite of rheological models, with particular emphasis on pressure- and rate-dependent formulations. Based on the generalized two-phase model developed by Khattri and Pudasaini [4], we analyse flow behaviour under varying shear rates, solid concentrations, and stress conditions. Our specific objectives are to: (1) characterize the rheological responses of different debris flow models, and (2) compare their predictive behaviour through simulation across key parameter regimes. By systematically analysing both classical and advanced rheological models, this study contributes to improved understanding of debris flow mechanics and supports the development of more reliable modelling frameworks for hazard assessment and infrastructure planning in landslide-prone regions.

2. Rheological models for geophysical flows

2.1. Newtonian and non-Newtonian model

Viscosity quantifies a fluid's resistance to flow caused by internal friction between its adjacent layers [42]. When a shear stress τ, is applied, the fluid deforms at a rate given by the strain rate $\dot{\gamma}$. This relationship is often described by the general *power-law* model [43]:

$$\tau = C\,(\dot{\gamma})^n\,, \tag{1}$$

$\dot{\gamma} = \dfrac{\partial u}{\partial y}$ is the strain rate (the rate of velocity change in the flow direction u to the normal coordinate y), C is the consistency index, and n is the flow behavior index that characterizes the fluid's rheological response [44, 45]. For Newtonian fluids ($n = 1$), the equation 1 reduces to:

$$\tau = \eta\dot{\gamma}, \tag{2}$$

with η as the constant dynamic viscosity. Newtonian fluids (e.g., water, air, ethanol) exhibit constant viscosity regardless of shear rate. In contrast, non-Newtonian fluids, where $n \neq 1$, show a non-linear relationship between stress and strain rate, and their apparent viscosity changes with deformation as in the Figure 1a.

2.2. Viscoplastic models

Many geophysical materials, including debris and mud flows, exhibit *viscoplastic behaviour*, where the material behaves as a rigid body below a threshold shear stress and flows like a viscous fluid beyond it.

2.2.1. Bingham plastic model

The Bingham plastic model [46, 47] describes materials that behave as rigid solids below a critical yield stress τ_y and flow as Newtonian fluids above it:

$$\dot{\gamma} = \begin{cases} 0 & \text{if } \tau < \tau_y, \\ \dfrac{\tau - \tau_y}{\eta} & \text{if } \tau \geq \tau_y, \end{cases} \tag{3}$$

where τ_y is the yield stress and η is the plastic viscosity.

The equation 3 means that the material is rigid for shear stress τ less than a threshold value τ_y, but it deforms as if it were a Newtonian viscous material once the yield strength is reached ($\tau \geq \tau_y$). Bingham plasticity model is widely applied to model materials such as toothpaste, drilling muds, fresh concrete, slurries, and lava flows, where flow occurs only after a critical yield stress is exceeded [48].

2.2.2. Herschel–Bulkley model

For materials exhibiting both a yield stress and shear-dependent viscosity, the Herschel–Bulkley model [13] provides a more accurate description:

$$\tau = \tau_y + C\,(\dot{\gamma})^n\,. \tag{4}$$

This model reduces to the Bingham plastic when $n = 1$ and to the power-law (Ostwal–de Waele) model when $\tau_y = 0$. The Herschel–Bulkley consistency index C and flow behavior index n, which are determined experimentally [13]. This model best describes complex fluids that transit from solid-like to liquid-like behavior under shear stress. One potential application of this model is the numerical simulation of fluid transport through pipelines for material conveyance in the agriculture, chemical, mining, dredging, oil, and gas sectors [49].

2.2.3. Quadratic rheological model for hyper-concentrated flows

For modeling sediment transport and hyper-concentrated flows, O'brien and Julien [50] introduced a quadratic rheological model:

$$\tau = \tau_y + \eta\dot{\gamma} + A_q\dot{\gamma}^2, \tag{5}$$

where τ is the total shear stress, τ_y is the total yield stress (potentially comprising cohesive and Mohr–Coulomb components), η is the dynamic viscosity, A_q is the quadratic stress coefficient, and $\dot{\gamma}$ is the shear rate. The model consists of three physically distinct stress contributions:

- a **yield stress** term (τ_y) representing the critical shear required to initiate flow,
- a **viscous stress** term ($\eta\dot{\gamma}$) describing linear Newtonian viscous dissipation, and
- a **quadratic stress** term ($A_q\dot{\gamma}^2$) capturing turbulent and particle dispersive stresses that dominate at higher shear rates [50].

This formulation extends traditional viscoplastic models by explicitly incorporating rate-dependent turbulent/dispersive effects, making it particularly suitable for high-sediment-concentration flows in geophysical and engineering contexts.

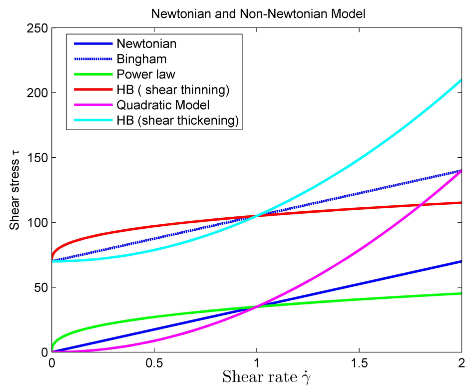
2.2.4. Bilinear model

Locat [51] introduced a model in which an apparent yield stress governs the transition from solid-like to fluid-like behavior under increasing shear stress as:

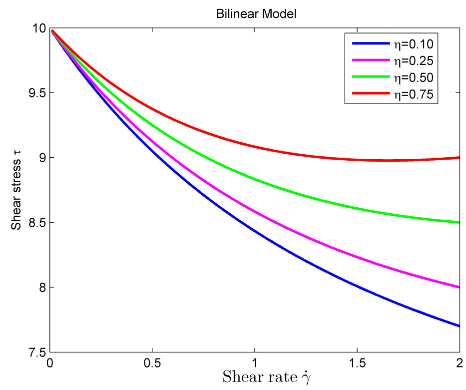
$$\tau = \tau_y + \eta\dot{\gamma} + \left(\frac{\tau_y \dot{\gamma}_0}{\dot{\gamma} + \dot{\gamma}_0}\right), \tag{6}$$

where τ_y is the apparent yield stress, η is the viscosity, $\dot{\gamma}$ is the shear rate, and $\dot{\gamma}_0$ is the shear rate at the transition from Newtonian to Bingham behavior.

The behavior of shear stress vs shear rate of different rheological models are presented in Figure (1a).



(a) Shear stress vs shear rate of different rheological models.



(b) Stress-strain response for varying viscosity.

Figure 1: Comparison of rheological model responses.

The simulations of the bilinear model has been presented in the Figure 1b. When the viscosity $\eta = 0.10\ \mathrm{m^2\,s^{-1}}$, the stress decreases rapidly as the shear rate increases. Similarly, while we increase the viscosity up to $\eta = 0.75\ \mathrm{m^2\,s^{-1}}$, the decent of shear stress rate gradually slows down.

2.3. Grain inertia and dilatancy

2.3.1. Bagnold rheology: Grain inertia and macroviscous regimes

The multiphase nature, along with the complex interaction between the phases in a debris flow, makes it difficult to characterize them using simple constitutive equations. Nevertheless, in many cases, the flow behavior of granular mixtures, especially those dominated by coarse particles, can be effectively described by Bagnold-type rheology [44, 45]. This formulation captures the essential physics of grain–grain interactions and distinguishes between two fundamental flow regimes: the *macroviscous regime* and the *grain-inertia regime*.

Bagnold [44] identified two distinct regimes in the rheology of granular suspensions:

- The *macroviscous regime* (viscous forces dominate), where shear stress scales linearly with shear rate, $\tau \propto \dot{\gamma}$.
- The *grain-inertial regime* (collisional stresses dominate), where shear stress scales quadratically, $\tau \propto \dot{\gamma}^2$.

These scalings correspond to exponents $n = 1$ and $n = 2$ in the Ostwald–de Waele power-law model [43]:

$$\tau = C\dot{\gamma}^n,$$

though Bagnold's original expressions included explicit dependencies on particle concentration and size. This behavior is characteristic of dense, high-shear flows in which momentum transfer occurs primarily through particle collisions rather than viscous dissipation [52].

2.3.2. Bagnold rheology and apparent viscosity

The grain-inertia rheological framework originally introduced by Bagnold [44] can be reformulated in terms of an *apparent viscosity* η_a [1, 53]:

$$\tau = \eta_a\,\dot{\gamma}, \quad \text{where} \quad \eta_a = C\,(\dot{\gamma})^{n-1}. \tag{7}$$

This formulation highlights how strongly sheared granular-fluid mixtures exhibit a nonlinear stress–strain relationship while retaining a formal analogy to Newtonian viscosity. The apparent viscosity η_a is not a material constant, but rather a shear-rate-dependent quantity that captures effects such as grain collisions, fluid mediation, and turbulence [1, 54]. Applications of this and related rheological frameworks to geophysical mass flows, including debris flows and rock-ice avalanches, are extensively discussed by Pudsaini [1], and Pudsaini and Mergili [53].

This formulation allows a unified interpretation: for $n = 1$, the fluid behaves as a classical Newtonian fluid with $\eta_a = C$. For $n = 2$, the apparent viscosity increases linearly with shear rate, reflecting *shear thickening* or *dilatant* behavior, as observed in coarse-particle-dominated flows. For $n < 1$, the viscosity decreases with shear rate, characterizing a *shear-thinning* regime.

2.3.3. Dilatancy and grain-inertia regimes in granular flows

A crucial mechanical feature of sheared granular materials in the grain-inertia regime is *dilatancy*—the volumetric expansion of a particle assembly under shear. This phenomenon, first systematically described by Reynolds [55], alters the pore geometry and, in saturated or partially saturated conditions, induces pore fluid pressure changes that modify the effective normal stress. The resulting increase in effective stress typically enhances frictional resistance, thereby influencing flow mobility [27, 56].

Dilatancy plays a particularly significant role in controlling flow dynamics, surge formation, and deposition patterns in geophysical mass movements such as debris avalanches and granular fronts [1, 57]. The transition between fluid-dominated (macroviscous) and particle-collision-dominated (inertial) regimes is commonly characterized by the dimensionless *Bagnold number* (Ba), originally introduced in the context of dispersed inertial flows [44]:

$$Ba = \frac{\rho d^2 \dot{\gamma}}{\eta}, \tag{8}$$

where ρ is the bulk density of the granular-fluid mixture, d is the mean particle diameter, $\dot{\gamma}$ is the shear rate, and η is the dynamic viscosity of the interstitial fluid.

- Low Bagnold numbers ($Ba \ll 1$) indicate a **viscous-dominated regime**, where fluid stresses mediate particle interactions and viscous dissipation dominates the energy budget.

- High Bagnold numbers ($Ba \gg 1$) correspond to an **inertia-dominated regime**, where grain–grain collisions govern momentum transfer and inertial stresses prevail [41, 54].

This dimensionless parameter provides a fundamental criterion for distinguishing flow regimes in granular–fluid mixtures and has been widely adopted in environmental and industrial granular flow modeling [58, 59].

2.4. Plasticity-based models

Plasticity-based models offer an alternative ways to describe the yielding behavior of geophysical flows, particularly those dominated by solid-phase interactions and frictional resistance [27, 31]. Unlike viscous or viscoplastic formulations, these models treat material failure as a function of shear and normal stresses [60], providing physically intuitive criteria for flow initiation and deformation in granular and rock-like materials [61, 62].

2.4.1. Mohr–Coulomb plasticity and hybrid viscous models

The *Mohr–Coulomb (M–C) plasticity model*, proposed for rapid granular flows by Savage and Hutter (1989) [31], defines failure based on the interaction between shear stress (τ) and normal stress (σ), according to which:

$$\tau = c + \sigma \tan \psi, \tag{9}$$

where c is the material cohesion and ψ is the internal friction angle. Yielding occurs when the Mohr circle representing the stress state touches or intersects this linear failure envelope. The main implication of Mohr's circle with the Coulomb stress of strength detects essential shear and normal stress pairs to predict failure planes in landslides or granular flows and extend to viscoplastic fluids with yield stress (e.g., magma and concrete).

2.4.2. Drucker–Prager yield criterion

To overcome the geometric limitations of the Mohr–Coulomb criterion in multidimensional applications, the Drucker–Prager (D–P) yield criterion offers a smooth, pressure-sensitive plasticity model [60]. It is a generalized form that approximates the Mohr–Coulomb surface in principal stress space using the second invariant of the deviatoric stress tensor II_{σ_d} and the mean pressure P:

$$\sqrt{II_{\sigma_d}} \geq \tau_y = \tau_p P, \tag{10}$$

where II_{σ_d} is the second invariant of the deviatoric stress tensor, P is the pressure (mean normal stress), and τ_p is a pressure-dependent friction coefficient related to the internal friction angle ψ.

The D–P model is particularly advantageous in numerical simulations such as finite element or finite volume methods, where smooth and differentiable yield surfaces are desirable [6, 14].

It is useful when simulating high-pressure, high-strain-rate regimes, such as those encountered during rapid runout or impact with terrain features.

2.5. Advanced models

2.5.1. $\mu(I)$-rheology for dense granular flows

A fundamental advancement in the continuum modeling of dry, dense granular flows was the introduction of the $\mu(I)$-rheology framework by the GDR MiDi consortium [26]. This approach characterizes the bulk mechanical response of granular materials through an effective friction coefficient μ_{eff} that depends on the dimensionless inertial number I:

$$\mu_{\text{eff}} = \mu(I) = \frac{\tau}{N}, \tag{11}$$

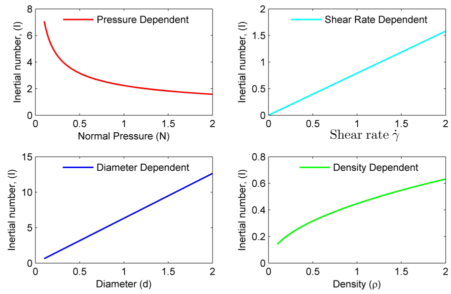


Figure 2: Variation of the inertial number for different parameters.

where τ is the shear stress and N is the normal (or confining stress) acting on the granular layer. The functional form $\mu(I)$ constitutes a nonlinear viscous constitutive law, as the resulting apparent granular viscosity depends on both the local strain rate and the normal pressure [40, 63].

The inertial number I is defined as a dimensionless strain rate that balances shear-induced fluctuations with confining pressure:

$$I = \frac{\dot{\gamma}\, d}{\sqrt{N/\rho}}, \tag{12}$$

where $\dot{\gamma}$ is the shear rate, d is the mean particle diameter, and ρ is the particle density. Physically, I distinguishes between quasi-static ($I \ll 1$), intermediate ($I \sim 1$), and collisional ($I \gg 1$) flow regimes [63], thereby unifying rate-independent plastic and rate-dependent viscous behaviors within a single constitutive framework.

Physically, the inertial number can be interpreted as the ratio of two timescales: the deformation timescale $T_\gamma = 1/\dot{\gamma}$ and the confinement timescale $T_N = d\sqrt{\rho/N}$:

$$I = \frac{T_N}{T_\gamma}. \tag{13}$$

This model has proven capable of capturing key characteristics of granular flows in various geometries such as inclined planes, annular shear cells, and rotating drums [24]. It explains transitions between quasi-static, dense, and collisional regimes, and describes both linear and Bagnold-type velocity profiles. We simulated $\mu(I)-$rheology with variation of the inertial number I with respect to different parameters as shown in the Figure 2. The behaviours of inertial number using the model 12 against normal load, shear rate, density, and diameter are illustrated in the Figure 2. The variation of $\mu(I)-$Rheology on the basis of the model equation 12 depending on different parameters. The inertial number I decreases non-linearly with normal pressure N, and increases weakly nonlinearly with the density ρ of the material, whereas it increases linearly with both the developed particle diameter d and shear rate $\dot{\gamma}$.

2.5.2. Pressure- and Rate-dependent Coulomb-viscoplastic model

To address the complex behavior of rapid dry granular flows, Domnik and Pudasaini [5] and Domnik et al. [6] proposed a pressure- and rate-dependent Coulomb-viscoplastic model

based on continuum mechanics for incompressible granular flows, governed by the mass and momentum conservation equations:

$$\nabla \cdot \mathbf{U} = 0, \tag{14}$$

$$\frac{D\mathbf{U}}{Dt} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{g}, \tag{15}$$

where $\mathbf{U}$ is the velocity vector, $\boldsymbol{\sigma}$ (source term of rheology) is the Cauchy stress tensor normalized by bulk density ρ, and $\mathbf{g} = (g\sin\zeta, -g\cos\zeta)^T$ denotes the gravitational acceleration in an inclined channel of angle ζ.

The constitutive relation for the stress tensor in viscoplastic form is:

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\nu\mathbf{d} + 2\tau_y\frac{\mathbf{d}}{||\mathbf{d}||}, \tag{16}$$

where p is the normalized pressure, ν is the kinematic viscosity, $\mathbf{d}$ is the strain-rate tensor defined by $\mathbf{d} = \frac{1}{2}\left[\nabla\mathbf{U} + (\nabla\mathbf{U})^T\right]$, and $||\mathbf{d}|| = \sqrt{2\,\mathrm{tr}(\mathbf{d}^2)}$ is its norm. The yield stress τ_y governs the transition between solid-like and fluid-like behavior.

The effective viscosity ν_{eff} is then given by:

$$\nu_{\text{eff}} = \nu + \frac{\tau_y}{||\mathbf{d}||}, \tag{17}$$

which leads to computational difficulties as $||\mathbf{d}|| \to 0$. To ensure numerical stability, Domnik et. al. [6] introduced a regularization term via an exponential cutoff parameter m_y:

$$\nu_{\text{eff}} = \nu + \frac{\tau_y}{||\mathbf{d}||}\left(1 - e^{-m_y||\mathbf{d}||}\right). \tag{18}$$

A key innovation of this model lies in its pressure-dependent yield stress:

$$\tau_y = \tau_p P, \tag{19}$$

where P is the pressure and τ_p is a material-specific friction parameter. This defines a Drucker–Prager-type yield condition [60]:

$$\sqrt{II_{\sigma_d}} \geq \tau_p P, \tag{20}$$

with II_{σ_d} being the second invariant of the deviatoric stress tensor. In 2D, this is equivalent to the Mohr–Coulomb yield condition with $\tau_p = \sin\psi$, where ψ is the internal friction angle, typically around 30° for natural granular materials. Moreover, the model can be extended to cohesive granular materials by introducing a cohesion term τ_c:

$$\tau_y = \tau_c + \tau_p P. \tag{21}$$

For dry granular materials, $\tau_c \approx 0$ [2].

2.5.3. Mixture bulk viscosity model

Khattri and Pudasaini [4, 64] introduced a physically consistent bulk viscosity model that extends the pressure- and rate-dependent viscoplastic framework initially proposed by Domnik and Pudasaini [5]. This model effectively captures the complex rheological behavior of multiphase mixtures composed of solid and fluid constituents by incorporating drift effects, pressure dependence, and solid concentration. The mixture bulk viscosity $\Lambda_{\eta m}$ is given by:

$$\Lambda_{\eta m} = \frac{1}{2}\left[\nu_s^e\alpha_s(\Lambda_u + \Lambda_w) + \nu_f\alpha_f(\lambda_u\Lambda_u + \lambda_w\Lambda_w)\right], \tag{22}$$

where ν_s^e is the effective solid-phase viscosity, ν_f is the fluid-phase kinematic viscosity, and α_s and $\alpha_f = 1-\alpha_s$ are the solid and fluid volume fractions, respectively. The terms Λ_u and Λ_w represent inertial drift coefficients in the x- and z-directions, defined as:

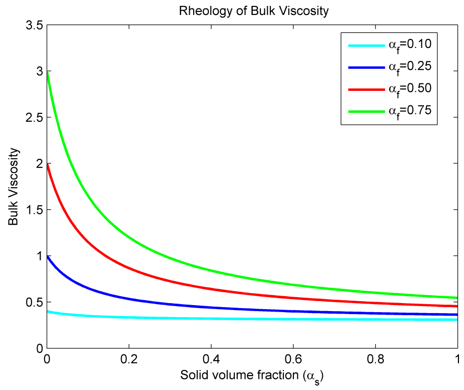


Figure 3: The relation between bulk viscosity and the solid volume fraction.

$$\Lambda_u = \frac{1}{\alpha_s + \lambda_u\alpha_f}, \tag{23}$$

$$\Lambda_w = \frac{1}{\alpha_s + \lambda_w\alpha_f}. \tag{24}$$

As shown in Figure 3, the parametric values used in the model 22 are $\nu_s^e = 0.30\,\mathrm{Kgm^{-1}s^{-1}}, \Lambda_u = 1, \Lambda_w = 1, 0 \leq \alpha_s \leq 1, \nu_f = 0.40\,\mathrm{m}^2 s^{-1}, \alpha_f = 0.10, 0.25, 0.50, 0.75$.

These drift-based coefficients effectively modify mass and momentum transport and are essential in characterizing the transition between dry granular and slurry-like flow regimes. As $\alpha_s \to 1$, the mixture rheology approaches that of a purely granular material, exhibiting pressure- and rate-dependent Coulomb-viscoplastic behavior. The effective viscosity of the solid phase is defined as:

$$\nu_s^e = \nu_s + \frac{\tau_{y_s}}{||\mathbf{d}_m||}\left(1 - e^{-m_y||\mathbf{d}_m||}\right), \tag{25}$$

where ν_s is the base solid-phase viscosity, τ_{y_s} is the yield stress, m_y is a regularization parameter, and $||\mathbf{d}_m||$ is the norm of the mixture strain-rate tensor. The yield stress is modeled as:

$$\tau_{y_s} = \tau_c + \tau_p\frac{p_m}{\Lambda_p}, \tag{26}$$

with τ_c representing cohesion, and $\tau_p p_m$ accounting for pressure-dependent yield. The mixture pressure and pressure drift factor are defined by:

$$p_m = (\alpha_s + \lambda_p\alpha_f)p_s, \tag{27}$$

$$\Lambda_p = \alpha_s + \lambda_p\alpha_f, \tag{28}$$

where p_s and p_f are the solid and fluid pressures, and λ_p is the pressure drift coefficient, satisfying $p_f = \lambda_p p_s$ [5, 6, 65, 66].

3. Discussion

Rheological variability in geophysical mass flows such as debris flows, avalanches, and mud flows is essential in explaining their dynamic behavior in response to changing stress and strain conditions. Our theoretical analysis and simulation of some rheological models, i.e., Bingham plastic, Herschel-Bulkley, quadratic, bilinear, Bagnold grain inertia, and pressure-dependent Coulomb-viscoplastic, reveal extreme mechanical responses that are important for explaining real-flow dynamics. In the study we found the following key results:

- Bingham plastic model (Figure 1a) exhibits a threshold yield stress (τ_y) below which the material behaves like a rigid solid. Above τ_y, the material behaves like a viscous fluid, and this renders it suitable for modeling mud flows and cohesive debris flows. Linear post-yield behavior is simpler to model computationally but cannot be expected to capture non-linear effects in very agitated flows.
- Herschel-Bulkley model (HB-model) extends the Bingham model to cover non-linear shear-thinning ($n < 1$) or shear-thickening ($n > 1$) behavior. Shear-thinning fluids (e.g., clay-rich debris flows) have decreasing viscosity with increasing shear rate, while shear-thickening suspensions (e.g., granular suspensions) become more resistant (see in Figure 1a). This is more suitable for natural flows where rheology is a function of shear rate.
- Quadratic model (Figure 1a) includes yield stress, viscous, and turbulent/dispersive stresses and is appropriate for hyper-concentrated flows. The turbulent component ($A_q\dot{\gamma}^2$) is important at large shear rates and simulates inertial granular collisions.
- Bilinear model (Figure 1b) resembles Newtonian for low shear rates but changes to Bingham-like behavior at a critical shear rate ($\dot{\gamma}_0$). Suitable for flows where weakly viscous fluid regions are low-stress areas that solidify under high stress.
- Bagnold grain-inertia model successfully describes the dilatant (shear-thickening) behavior in granular flow, for which stress goes quadratically with shear rate. Suitable for coarse-grained flows and high shear rates but may fail under slow-moving, dense debris where frictional contacts dominate.
- Pressure-dependent Coulomb-viscoplastic model has a pressure-dependent yield criterion ($\tau_y = \tau_p P$), which is consistent with Mohr-Coulomb failure mechanics. It simulates granular flows more realistically, including shear banding, plug formation, and solid-fluid transitions. Our calculations suggest that increased solid volume fraction (α_s) is more favorable to stratification, with increased viscosity near the base and decreased viscosity at the free surface.

4. Conclusion and future works

This study emphasizes the importance of rheological variation in geophysical mass flow behavior prediction. While easier models (e.g., Bingham, Herschel-Bulkley) offer convenient simplifications, more advanced formulations (e.g., pressure-dependent viscoplasticity) demonstrate more mechanistic understanding. The findings emphasize the reality that any single rheology will not be applicable everywhere; instead, model selection must be commensurate with flow composition, stress regime, and spatial-temporal scales. Subsequent research must be focused on multi-stage interactions, particle-size influences, and appropriate rheological adjustment to advance hazard evaluation and engineering solutions.

We focused on different rheological aspects of geophysical mass flows. The relationship between shear stress and induced shear rate showed significant variations for different compositions of the mixture material. Mixture rheology revealed non-linear behaviour with increasing solid volume fractions for different pressures. Natural debris flows, however, exhibit a wide range of features in many complex natural situations. The theory can yet be improved upon in many ways. Basic themes that need to be investigated include how to modify the equation of flow-age to account for the complex features of natural debris flow, such as the mixing of a wide range of particle sizes, non-homogeneity in concentration, particle rotation, and flow instability in extremely steep channels.

Most models account for homogeneous mixtures, whereas natural debris flows are polydisperse grains and variable fluid content. Three-dimensional effects and non-local rheology (e.g., $\mu(I)-$rheology) need to be accounted for in future works in the case of complex terrain interactions. Field and large-scale simulations would be needed to calibrate parameters for geotechnical and industrial applications.

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