Kathmandu University Journal of Science, Engineering and Technology, Vol. 18, No. 2, December 2024

Kathmandu University
Journal of Science, Engineering and Technology

A study of common fixed point of enriched contractions

O. J. Omidire*[a], K. R. Tijani[a], and B. T. Ishola[b]

[a] Department of Mathematical Sciences, Osun State University, Osogbo, Nigeria.
[b]Department of Mathematics, Obafemi Awolowo University, Ile- Ife, Nigeria.

Abstract

In this paper, we study convergence of Jungck-Schaefer iterative scheme to the common fixed point of generalized enriched contractions. Some novel general class of enriched contractive definitions called *enriched-Jungck contractions* are presented and we study the existence and uniqueness of common fixed points for these class of mappings in Banach spaces using Jungck-Schaefer iterative techniques. Our results unify, generalize and extend some recently announced related results in literature.

Keywords: Generalized enriched Jungck-contraction; Jungck-Schaefer iteration; Positively homogeneous function; Common fixed point and Banach spaces.

1. Introduction

Given a complete metric space (M, d) and a self-mapping P on M such that:

$$d(Pu, Pv) \leq ad(u, v) \ \forall \, u, \ v \in M, \ \ a \in [0, 1) \ \ fixed. \tag{1.1}$$

The operator P in (1.1) above is called an a-contraction (or Banach contraction). Banach in his celebrated result proved that Picard iteration converges to the unique fixed point of P in M see [1].

Motivated by Banach's work, Rakotch [2], generalized Banach's assertion by introducing a monotone decreasing function $\alpha : (0, \infty) \to [0, 1)$ such that, for each $u, v \in B, u \neq v$,

$$d(Gu, Gv) \leq \alpha(d(u, v)) \tag{1.2}$$

Kannan [3] claimed that G need not be continuous to have fixed point, but compensated for this using the following more robust contraction definitions: There exists $a \in [0, \frac{1}{2})$ such that

$$d(Gu, Gv) \leq a[d(u, Gu) + d(v, Gv)], \ \ \forall \, u, \ v \in B. \tag{1.3}$$

Over the years, there have been several generalizations and extensions of classical Banach's fixed point theorem.

Jungck [4] moved a step further by introducing the notion of common fixed point of mappings $S, G : M \to M$ defined on a complete metric space (M, d). He employed the contractive definition below:

For $S, G : M \to M$, there exists $a \in (0, 1)$ such that,

$$d(Gu, Gv) \leq ad(Su, Sv), \ \ \forall \, u, v \in M, \tag{1.4}$$

In generalizing inequalities (1.1) - (1.3) above and many more related results in literature, Akram et. al [5] gave the definition below:

Definition 1.1. [5]: "A self-map T of a metric space X is called an A-contraction if:

$$d(Tx, Ty) \leq \alpha(d(x, y), d(x, Tx), d(y, Ty))$$

for all $x, y \in X$ and some $\alpha \in (A)$, where (A) is the set of all functions $\alpha : \mathbb{R}^3_+ \to \mathbb{R}_+$ satisfying:

i) α is continuous on the set $\mathbb{R}^3_+$ (with respect to Euclidean metric on $\mathbb{R}^3$);

ii) if any of the conditions $a \leq \alpha(a, b, b)$, or $a \leq \alpha(b, b, a)$, or $a \leq \alpha(b, a, b)$ holds for some $a, b \in \mathbb{R}_+$, then there exists $k \in [0, 1)$ such that $a \leq kb$."

Olatinwo and Omidire [6] extended results in [1] and [5] by proving some common fixed point theorems for the below general class of mapping:

Definition 1.2. [6]: "Let (X, d) be a metric space and T, S :

*Corresponding author. Email: olaoluwa.omidire@uniosun.edu.ng

$X \to X$ such that

$$d(Tx, Ty) \leq \varphi(d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), [d(Sx, Tx)]^r [d(Sy, Tx)]^p d(Sx, Ty), d(Sy, Tx)[d(Sx, Tx)]^m) \quad \forall\, x,\ y \in X; r, p, m \in \mathbb{R}_+ \tag{1.5}$$

and that (1.5) is satisfied by the set of all functions
$\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ such that:

(i) φ is continuous on the set $\mathbb{R}_+^5$ (with respect to Euclidean metric on $\mathbb{R}^5$);

(ii) if any of the conditions $a \leq \varphi(a, b, b, b, b)$, or $a \leq \varphi(b, b, a, b, b)$, or $a \leq \varphi(b, b, a, c, c)$ holds for some $a, b, c \in \mathbb{R}_+$, then there exists a constant $k \in [0, 1)$ such that $a \leq kb$.

However, Berinde and Pacurar [7] introduced the concept; enrichment of non-linear mappings, called an enriched contractions which includes, amongst many others; a-contraction.

Definition 1.3. [7]: "Let $(Y, ||.||)$ be a normed linear space. An operator $P : Y \to Y$ is called enriched contraction if $\exists\, c \in [0, \infty),\ and\ \beta \in [0, c+1)$ such that

$$||c(x - y) + Tx - Ty|| \leq \beta||x - y||, \forall\, x, y\ \in Y \text{"} \tag{1.6}$$

In [7], it was shown that any contractive condition (1.6) reduced to (1.1) if $c = 0$. (See example 1 of [7] for details).

Definition 1.4. [4]: Let M be a complete metric space, and suppose $G, U : M \to M$. For $v_0 \in M$, sequence $\{Uv_n\}_{n=0}^{\infty} \subset M$ generated by

$$Uv_{n+1} = Gv_n,\ n \geq 0,$$

is called Jungck's iterative process.

Using idea of Jungck, many authors have improved on the existing iterative techniques.

Definition 1.5. [8, 9]: "Let B be a Banach space, and the pair of operators $U, G : B \to B$. For any $v_0 \in B$, the sequence $\{Uv_n\}_{n=0}^{\infty}$, defined by

$$Uv_{n+1} = (1 - c)Uv_n + cGv_n, \quad n \geq 0, \quad c \in (0, 1). \tag{1.7}$$

is called Jungck-Schaefer iteration."

For more on Jungck-type iterative algorithms, interested reader can see [8, 10, 11, 12, 13, 14] and references therein.

In this paper, we give extensions of the celebrated results of Jungck [4] using enriched contraction definitions, recently announced in [7] in line with generalizations given in the papers [6] and [5] by presenting some general class of enriched contractive definitions called *enriched-Jungck contractions* and study the existence and convergence of Jungck-Schaefer iterative techniques (as introduced in [14]) to a unique common fixed point of these class of mappings satisfying commuting and compatible conditions.

The following are vital tools in obtaining our results:

Definition 1.6. [15]: "Consider a function ψ: $\mathbb{R}_+ \to \mathbb{R}_+$ satisfying:

(a) ψ is monotone increasing i.e $t_1 \leq t_2$ $\implies \psi(t_1) \leq \psi(t_2)$;

(b) $\psi^n(t)$ converges to 0 as $n \to \infty$ for all $t \in \mathbb{R}_+$;

(c) $\sum_{n=0}^{\infty} \psi^n(t)$ converges for all $t \geq 0$."

Remark 1.7. (i) A function ψ satisfying (a) and (b) in definition 1.6 above is said to be a comparison function.

(ii) A function ψ satisfying (a) and (c) in definition 1.6 above is said to be a (c)- comparison function.

(iii) Any comparison function satisfies $\psi(0) = 0$.

2. Preliminary results

Definition 2.1. Let $(L, ||.||)$ be a normed linear space. An Operator $P : L \to L$ is said to be a generalized enriched Jungck-contraction if for any $c \in [0, \infty)$ and a function ϕ with $\phi(t) \in [0, c+1)$, there is a map $Q : L \to L$, such that $\forall\, x, y\ \in L$ we have

$$||c(Qx - Qy) + Px - Py|| \leq \phi[||Qx - Qy||, ||Qx - Px||, ||Qy - Py||, (||Qx - Px||)^r (||Qy - Px||)^p (||Qx - Py||), ||Qy - Px|| (||Qx - Px||)^m], \quad \forall\ x, y \in L, r, m, p\ \in \mathbb{R}_+. \tag{2.1}$$

and ϕ is a function defined by $\phi : \mathbb{R}_+^5 \to \mathbb{R}_+$ such that:

(i) ϕ is continuous on the set $\mathbb{R}_+^5$ (with respect to Euclidean metric on $\mathbb{R}^5$);

(ii) if any of the conditions $f \leq \phi(f, g, g, g, g)$, or $f \leq \phi(g, g, f, g, g)$, or $f \leq \phi(g, g, f, h, h)$ holds for some f, g, h in $\mathbb{R}_+$, then there exists a constant $k \in (0, 1)$ such that $f \leq k(g)$.

The next definition is a generalization of Definition 2.1 using a c-comparison function (Definition 1.6).

Definition 2.2. Let $(L, ||.||)$ be a normed linear space. A mapping $P : L \to L$ is said to be a generalized enriched $\psi-$ Jungck-contraction if for any $c \in [0, \infty)$, and a function ϕ with $\phi(t) \in [0, c+1)$, there is a map $Q : L \to L$, such that $\forall\, x, y\ \in L$ we have

$$||c(Qx - Qy) + Px - Py|| \leq \phi[||Qx - Qy||, ||Qx - Px||, ||Qy - Py||, (||Qx - Px||)^r (||Qy - Px||)^p (||Qx - Py||), ||Qy - Px|| (||Qx - Px||)^m], \quad \forall\ x, y \in L, r, m, p\ \in \mathbb{R}_+. \tag{2.2}$$

and ϕ is a function defined by $\phi : \mathbb{R}_+^5 \to \mathbb{R}_+$ such that:

(i) ϕ is continuous on the set $\mathbb{R}_+^5$ (with respect to Euclidean metric on $\mathbb{R}^5$);

(ii) if any of the conditions $f \leq \phi(f, g, g, g, g)$, or $f \leq \phi(g, g, f, g, g)$, or $f \leq \phi(g, g, f, h, h)$ holds for some f, g, h in $\mathbb{R}_+$, then there exists a positively homogeneous c-comparison function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f \leq \psi(g)$.

Definition 2.3. Let $(L, ||.||)$ be a normed linear space. A mapping $P : L \to L$ is said to be a generalized enriched Jungck-contraction if for any $c \in [0, \infty)$ and a function ϕ

with $\phi(t) \in [0, c+1)$, there is a mapping $Q : L \to L$, such that $\forall\, u, v \in L$ we have

$$||c(Qu - Qv) + Pu - Pv|| \leq \phi[||Qu - Qv||, ||Qu - Pu||, ||Qv - Pv||], \quad \forall\ u, v \in L.$$

and ϕ is a function defined by $\phi : \mathbb{R}^3_+ \to \mathbb{R}_+$ such that:

(i) ϕ is continuous on the set $\mathbb{R}^3_+$ (with respect to Euclidean metric on $\mathbb{R}^3$);

(ii) if any of the conditions $f \leq \phi(f, g, g)$, or $f \leq \phi(g, f, g)$, or $f \leq \phi(g, g, f)$ holds for some f, g in $\mathbb{R}_+$, then there exists a constant $k \in (0, 1)$ such that $f \leq k(g)$.

Definition 2.4. Let $(L, ||.||)$ be a normed linear space. A mapping $P : L \to L$ is said to be an enriched $\psi-$ Jungck contraction if for any $c \in [0, \infty)$, and function ϕ with $\phi(b) \in [0, c+1)$, there is a map $Q : L \to L$, such that $\forall\, u, v \in L$ we have

$$||c(Qu - Qv) + Pu - Pv|| \leq \phi[||Qu - Qv||, ||Qu - Pu||, ||Qv - Pv||], \quad \forall\ u, v \in L.$$

and ϕ is a function defined by $\phi : \mathbb{R}^3_+ \to \mathbb{R}_+$ such that:

(i) ϕ is continuous on the set $\mathbb{R}^3_+$ (with respect to Euclidean metric on $\mathbb{R}^3$);

(ii) if any of the conditions $f \leq \phi(f, g, g)$, or $f \leq \phi(g, f, g)$, or $f \leq \phi(g, g, f)$ holds for some f, g in $\mathbb{R}_+$, then there exists a positively homogeneous c-comparison function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f \leq \psi(g)$.

Example 2.5. Let $X = [0, 2]$ be endowed with the usual norm. And let P, Q be self maps on X, defined as $P(u) = 2u^2 + u;\ \ Q(u) = u,\ \forall\, u \in X$.

We have,

$$\begin{aligned} |Pu - Pv| &= |(2u^2 + u) - (2v^2 + v)| \\ &= |2u^2 - 2v^2 - v + u| \\ &= |2(u^2 - v^2) + (u - v)| \\ &= |2(u + v)(u - v) + (u - v)| \\ &= |(2(u + v) + 1)(u - v)| \\ &= (2(u + v) + 1)|(u - v)| \\ &= (2(u + v))|Qu - Qv|. \end{aligned}$$

Clearly, for all $u, v \in X, \quad |Pu - Pv| \geq |Qu - Qv|$.

Hence, P with respect to Q is not a Jungck contraction.

But, P is a generalized enriched Jungck contraction; as shown below:

Choose $c = 1,\ \ r = p = m = 0$ and define ϕ as

$$\phi(a, b, c, d, e) = a + b + c + d + e, \quad \forall\ a, b, c, d, e\ \in\ \mathbb{R}_+.$$

Then

$$\begin{aligned} |c(Qu - Qv) \\ +Pu - Pv| &= |Qu - Qv + Pu - Pv| \\ &= |Pu - Qv + Qu - Pv| \\ &\leq |Pu - Qv| + |Qu - Pv| \\ &\leq |Pu - Qv| + |Qu - Pv| + \\ &\quad\ |Qu - Qv| + |Qu - Pu| + |Qv - Pv| \\ &\leq \phi(|Pu - Qv|, |Qu - Pv|, \\ &\quad\ |Qu - Qv|, |Qu - Pu|, |Qv - Pv|). \end{aligned}$$

That is, P with respect to Q is a generalized enriched Jungck contraction.

Remark 2.6. (i) If $c = 0$, Definition 2.1 above reduces to Definition 1.2 (with $d(x, y) = ||x - y||$), see [6]

(ii) If $\psi = k$ (a constant) then Definition 2.2 becomes Definition 2.1.

Remark 2.7. Let X be a convex subset of a linear space L and P a self map on X. If there is an identity map $Q : X \to X$. Then for any $\lambda \in (0, 1)$, the set of all fixed points of a mapping $P_\lambda \colon X \to X$ given by $P_\lambda(x) = (1 - \lambda)Qx + \lambda Px$ coincides with $Fix(Q)$. Also the set of all fixed points of a mapping

$$(P_\lambda, Q)\colon X \to X$$

given by Jungck-Schaefer iterative sequence

$$Qu_{n+1} = (1 - \lambda)Qu_n + \lambda Pu_n, \tag{2.3}$$

coincides with Jungck iteration

$$Qu_{n+1} = P_\lambda u_n,\ n \geq 0,\ i.e\ Fix(P_\lambda) = Fix(Q).$$

Lemma 2.8. *(Analogue of Jungck's fixed point theorem) Supposing D is a nonempty, closed subset of a Banach space B, and let P be a mapping from D to D. If there exists a continuous, selfmap Q on D which commutes with P and $P(D) \subset Q(D)$ satisfies*

$$||Px - Py|| \leq k||Qx - Qy||,\ \forall\ x, y \in D,\ k \in [0, 1) \tag{2.4}$$

Then, P and Q have a unique common fixed point in D.

The following definitions and results shall be required in Section 4.

Let $(N, ||\cdot||)$ be a normed linear space.

Definition 2.9. [16] "Two self-mappings P and Q on X are weakly commuting if

$$||PQx - QPx|| \leq ||Px - Qx||,\ \forall\, x \in X.$$

Definition 2.10. [17] "Self mappings P and Q on X are compatible if and only if

$$\lim_{n\to\infty} ||PQx_n - QPx_n|| = 0$$

whenever $\{x_n\}$ is a sequence in X, such that

$$\lim_{n\to\infty} P(x_n) = \lim_{n\to\infty} Q(x_n) = w$$

for some $w \in X$.

Remark 2.11. (i) Definition (2.10) was originally given in metric space settings. Since metric is induced by the norm (i.e $d(x, y) = ||x - y||$), it is adapted to a normed space settings.
(ii) Commuting mappings are weakly commuting and the reverse is not true, see [17] for example.

(iii) Weakly commuting mappings are compatible, but compatible mappings may not be weakly commuting see [17] for illustration.

Lemma 2.12. *[16, 17]: "Let P and Q be two compatible self-mappings on Banach space B.*

- If $Px^* = Qx^*$, then $PQx^* = QPx^*$.
- Assume that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n = w$ for some $w \in B$.
  (a) If P is continuous at w, then
  $$\lim_{n\to\infty} QPx_n = Pw.$$
  If Q is continuous at w, then
  $$\lim_{n\to\infty} PQx_n = Qw.$$
  (b) If P and Q are continuous at w, then
  $$Pw = Qw \quad \text{and} \quad QPw = PQw.$$

Theorem 2.13. *[16] "Let $(B, ||\cdot||)$ be a Banach space and $P, Q : B \to B$ be two mappings for which exist $c \in (0, +\infty]$ and $\theta \in [0, c+1)$, such that*

$$||c(u-v) + Pu - Pv|| \leq \theta||Qu - Qv||, \forall\ u, v\ \in B. \tag{2.5}$$

If the below conditions are satisfied:

- P_λ and Q are compatible mappings, where
  $$P_\lambda(u) = (1-\lambda)u_n + \lambda Pu_n,\ \ n \geq 0,$$
- P_λ and Q are continuous,

then

- $Fix(P) = Fix(Q) = \{x\}$;
- there exists $\lambda \in (0, 1]$, so that the iterative sequence $\{Qu_{n+1}\}_{n=0}^{\infty}$ converges strongly to x."

Remark 2.14. (i) Definition 2.3 generalizes 2.5.

i.e, if $\phi[||Qx-Qy||, ||Qx-Px||, ||Qy-Py||] = \theta||Qx-Qy||$ then Definition 2.3 reduces to inequality 2.5.

(ii) If $\phi[||Qx - Qy||, ||Qx - Px||, ||Qy - Py||, (||Qx - Px||)^r(||Qy - Px||)^p(||Qx - Py||), ||Qy - Px||(||Qx - Px||)^m] = \theta||Qu-Qv||$, Definition 2.1 reduces to inequality 2.5.

3. Main results

Theorem 3.1. *Let $(B, ||.||)$ be a Banach space and $P, Q : B \to B$ be commuting mappings satisfying Definition 2.1. If Q is continuous and $P(B) \subseteq Q(B)$, then:*

(i) P_λ and Q have a unique common fixed point $u^* \in B$;

(ii) There exists $\lambda \in (0, 1]$ such that the Jungck-Schaefer iteration $\{Qu_n\}_{n=0}^{\infty}$, defined by 2.3 converges to u^*, for any $u_0 \in B$.

Proof: *Since $c \geq 0$, there are two possible cases (i.e $c = 0$ and $c > 0$).*

Case 1: *For $c = 0$, inequality 2.1 reduces to 1.5 of the author in [6] and the prove follows the same argument of Theorem (2.1) in [6].*

Case 2: *When $c > 0$. Considering sequence defined by 2.3 and for $\lambda = \frac{1}{c+1}$, we have that*

$$c = \frac{1-\lambda}{\lambda}, \tag{3.1}$$

then $\forall\ u, v\ \in B$, inequality 2.1 becomes

$$\begin{aligned} &||\frac{1-\lambda}{\lambda}(Qu - Qv) \\ &\quad + Pu - Pv|| \leq \phi[||Qu - Qv||, ||Qu - Pu||, \\ &\qquad ||Qv - Pv||, (||Qu - Pu||)^r \\ &\qquad (||Qv - Pu||)^p(||Qu - Pv||), \\ &\qquad ||Qv - Pu||(||Qu - Pu||)^m] \end{aligned}$$

$$\begin{aligned} ||1 - \lambda(Qu - Qv) \\ + \lambda Pu - \lambda Pv|| &= ||(1-\lambda)Qu + \lambda Pu \\ &\quad -(1-\lambda)Qv + \lambda Pv|| \\ &= ||Qu_{n+1} - Qv_{n+1}|| \\ &= ||P_\lambda u_n - P_\lambda v_n|| \\ &\leq \lambda\Big(\phi[||Qu - Qv||, ||Qu - Pu||, \\ &\quad ||Qv - Pv||, \\ &\quad (||Qu - Pu||)^r(||Qv - Pu||)^p \\ &\quad (||Qu - Pv||), \\ &\quad ||Qv - Pu||(||Qu - Pu||)^m]\Big). \end{aligned} \tag{3.2}$$

Now, considering Jungck-Schaefer iterative process $\{Qu_n\}_{n=0}^{\infty}$ defined by (2.3), which actually coincides with Jungck iteration associated with P_λ and Q i.e.

$$Qu_n = P_\lambda u_{n-1},$$

let $v = u_n$ and $u = u_{n-1}$, so

$$\begin{aligned} ||Qu_n - Qu_{n+1}|| &= ||P_\lambda u_{n-1} - P_\lambda u_n|| \\ &\leq \lambda\Big(\phi[||Qu_{n-1} - Qu_n||, \\ &\quad ||Qu_{n-1} - P_\lambda u_{n-1}||, ||Qu_n - P_\lambda u_n||, \\ &\quad (||Qu_{n-1} - P_\lambda u_{n-1}||)^r \\ &\quad (||Qu_n - P_\lambda u_{n-1}||)^p \\ &\quad (||Qu_{n-1} - P_\lambda u_n||), \\ &\quad ||Qu_n - P_\lambda u_{n-1}|| \\ &\quad (||Qu_{n-1} - P_\lambda u_{n-1}||)^m]\Big) \\ &= \lambda\Big(\phi[||Qu_{n-1} - Qu_n||, ||Qu_{n-1} - Qu_n||, \\ &\quad ||Qu_n - Qu_{n+1}||, \\ &\quad (||Qu_{n-1} - Qu_n||)^r(||Qu_n - Qu_n||)^p \\ &\quad (||Qu_{n-1} - Qu_{n+1}||), \\ &\quad ||Qu_n - Qu_n||(||Qu_{n-1} - Qu_n||)^m]\Big) \\ &= \lambda\Big(\phi[||Qu_{n-1} - Qu_n||, ||Qu_{n-1} - Qu_n||, \\ &\quad ||Qu_n - Qu_{n+1}||, 0, 0]\Big) \\ &\leq \lambda \times k\Big(||Qu_{n-1} - Qu_n||\Big), \end{aligned} \tag{3.3}$$

i.e

$$||Qu_n - Qu_{n+1}|| \leq \mu(||Qu_{n-1} - Qu_n||), \tag{3.4}$$

where $\mu = \lambda \times k < 1$.

Inductively from (3.4) we have

$$\begin{aligned} ||Qu_n - Qu_{n+1}|| &\leq \mu(||Qu_{n-1} - Qu_n||) \\ &\leq \mu^2(||Qu_{n-2} - Qu_{n-1}||) \\ &\leq \mu^3(||Qu_{n-3} - Q_{n-2}||) \\ &\leq \mu^n(||Qu_0 - Qu_1||), \end{aligned}$$

i.e

$$||Qu_n - Qu_{n+1}|| \leq \mu^n(||Qu_0 - Qu_1||). \tag{3.5}$$

By repeated application of triangle inequality on (3.5), for any $p \in \mathbb{N}$ we have:

$$\begin{aligned} ||Qu_n - Qu_{n+p}|| \leq\ & \frac{\mu^n(1-\mu^p)}{1-\mu} \\ & (||Qu_0 - Qu_1||) \to 0, \\ & as\, n \to \infty \ (0 \leq \mu = \lambda \times k < 1) \end{aligned} \tag{3.6}$$

Hence, $\{Qu_n\}_{n=0}^{\infty}$ is a Cauchy sequence in Banach space B, then, there exists $u^* \in B$ such that

$$\lim_{n\to\infty} Qu_n = \lim_{n\to\infty} P_\lambda u_{n-1} = u^*.$$

With continuity of Q and commutativity of P and Q, we have the following:

$$Qu^* = Q(\lim_{n\to\infty} Qu_n) = \lim_{n\to\infty} S^2 u_n \tag{3.7}$$

$$Qu^* = Q(\lim_{n\to\infty} T_\lambda u_n) = \lim_{n\to\infty}(QP_\lambda u_n) = \lim_{n\to\infty}(P_\lambda Q u_n). \tag{3.8}$$

Thus, with $u_n = Qu_n$ and $v_n = u^*$ in inequality (3.2) we have

$$\begin{aligned} ||P_\lambda(Qu_n) - P_\lambda u^*|| \leq& \lambda\Big(\phi[||Q^2u_n - Qu^*||, \\ & ||Q^2u_n - P_\lambda(Qu_n)||, ||Qu^* - P_\lambda u^*||, \\ & (||Q(Qu_n) - P_\lambda(Qu_n)||)^r \\ & (||Qu^* - P_\lambda(Qu_n)||)^p \\ & (||Q(Qu_n) - P_\lambda u^*||), \\ & ||Qu^* - P_\lambda(Qu_n)|| \\ & (||Q(Qu_n) - P_\lambda(Qu_n)||)^m]\Big). \\ =& \lambda\Big(\phi[||Q^2u_n - Qu^*||, \\ & ||Q^2u_n - P_\lambda(Qu_n)||, ||Qu^* - P_\lambda u^*||, \\ & (||Q^2u_n - P_\lambda(Qu_n)||)^r \\ & (||Qu^* - P_\lambda(Qu_n)||)^p \\ & (||Q^2u_n - P_\lambda u^*||), \\ & ||Qu^* - P_\lambda(Qu_n)|| \\ & (||Q^2u_n - P_\lambda(Qu_n)||)^m]\Big). \end{aligned} \tag{3.9}$$

Applying (3.7) and (3.8) into (3.9), as $n \to \infty$ gives

$$\begin{aligned} ||Qu^* - P_\lambda u^*|| \leq& \lambda\Big(\phi[||Qu^* - Qu^*||, \\ & ||Qu^* - Qu^*||, ||Qu^* - P_\lambda u^*||, \\ & (||Qu^* - Qu^*)||)^r \\ & (||Qu^* - Qu^*)||)^p(||Qu^* - P_\lambda u^*||), \\ & ||Qu^* - Q_u^*||(||Qu^* - Qu^*)||)^m]\Big) \\ =& \lambda\Big(\phi[0, 0, ||Qu^* - P_\lambda u^*||, 0, 0]\Big) \\ \leq& \lambda \times k(0) = 0. \end{aligned}$$

Therefore, we have $Qu^* = P_\lambda u^*$.

And again, with $v_n = u^*$ in inequality (3.2), we also have

$$\begin{aligned} ||P_\lambda u_n - P_\lambda u^*|| \leq& \lambda\Big(\phi[||Qu_n - Qu^*||, \\ & ||Qu_n - P_\lambda u_n)||, ||Qu^* - P_\lambda u^*||, \\ & (||Qu_n - P_\lambda u_n)||)^r \\ & (||Qu^* - P_\lambda u_n)||)^p(||Qu_n - P_\lambda u^*||), \\ & ||Qu^* - P_\lambda u_n||(||Qu_n - P_\lambda u_n||)^m]\Big). \end{aligned}$$

Taking limit as $n \to \infty$ gives

$$\begin{aligned} ||u^* - P_\lambda u^*|| \leq& \lambda\Big(\phi[||u^* - Qu^*||, \\ & ||u^* - u^*||, ||Qu^* - P_\lambda u^*||, \\ & (||u^* - u^*||)^r(||Qu^* - u^*||)^p \\ & (||u^* - P_\lambda u^*||), ||Q_\lambda u^* - u^*|| \\ & (||u^* - u^*)||)^m]\Big) \\ =& \lambda\Big(\phi[||u^* - P_\lambda u^*||, 0, 0, 0, 0]\Big) \\ \leq& \lambda \times k(0) \\ =& 0. \end{aligned}$$

This implies that, $u^* = P_\lambda u^*$.

Hence,

$$Qu^* = P_\lambda u^* = u^*.$$

Now, we prove the uniqueness of this common fixed point. Suppose not, then there exists $u^* \in B$, such that $Qu^* = P_\lambda u^* = u^*, Qv^* = P_\lambda v^* = v^*$,

$$\begin{aligned} ||u^* - v^*|| =& ||P_\lambda u^* - P_\lambda v^*|| \\ \leq& \lambda\Big(\phi[||Qu^* - Qv^*||, \\ & ||Qu^* - P_\lambda u^*||, ||Qv^* - P_\lambda v^*||, \\ & (||Qu^* - P_\lambda u^*||)^r(||Qv^* - P_\lambda u^*||)^p \\ & (||Qu^* - P_\lambda v^*||), \\ & ||Qv^* - P_\lambda u^*||(||Qu^* - P_\lambda u^*||,)^m]\Big) \\ =& \lambda\Big(\phi[||u^* - v^*||, 0, 0, 0, 0]\Big) \\ \leq& \lambda \times k(0) \\ =& 0. \end{aligned}$$

Therefore, $u^* = v^*$

Example 3.2. Let $B = [-2, 0]$ be endowed with usual norm and P, Q be self-mappings on B defined as

$$P(u) = u^2 + 2u \text{ and } Q(u) = u.$$

It is clear to see that

$$QP(u) = PQ(u) = u^2+2u,\ \forall\, u \in B,\ \text{ and } P(B) \subseteq Q(B).$$

We show that P with respect to Q does not satisfy Jungck condition as we have below:

$$\begin{aligned}
|Pu - Pv| &= |(u^2 + 2u) - (v^2 + 2v)| \\
&= |u^2 - v^2 - 2v + 2u| \\
&= |(u^2 - v^2) + 2(u - v)| \\
&= |(u + v)(u - v) + 2(u - v)| \\
&= |((u + v) + 2)(u - v)| \\
&= ((u + v) + 2)|(u - v)| \\
&= (2 + u + v))|Qu - Qv|.
\end{aligned}$$

Clearly, for all $u, v \in B$, $\quad |Pu - Pv| \geq |Qu - Qv|$.

Hence, P with respect to Q is not a Jungck contraction.

However, define ϕ as

$$\phi(a, b, c, d, e) = a + b + c + d + e, \quad \forall\ a, b, c, d, e\ \in\ \mathbb{R}_+,$$

and choose $c = 1$, then by following similar argument in Example 2.5, it easy to see that P with respect to Q satisfies inequality 2.1. that is,

$$\begin{aligned}
||c(Qu - Qv) \\
+Pu - Pv|| \quad &\leq \quad \phi\Big[||Pu - Qv||, \\
&||Qu - Pv||, ||Qu - Qv||, \\
&||(Qu - Pu)^r(||Qv - Pu||)^p \\
&(||Qu - Pv||)||, \\
&||Qv - Pu||(||Qu - Pu||)^m\Big].
\end{aligned}$$

And all conditions of Theorem 3.1 are met. Hence, common fixed point of P and Q exists. Indeed,

$$Fix(P) = Fix(Q) = -1.$$

Also, with $\lambda = \frac{1}{2}$, and $u_0 = -2$, Jungck-Schaefer iteration 2.3 converges to the unique fixed point of P and Q

Theorem 3.3. *Let $(B, ||.||)$ be a Banach space and $P, Q : B \to B$ be commuting and generalized enriched $\psi-$Jungck contraction. If Q is continuous and $P(B) \subseteq Q(B)$. Then:*

(i) P_λ and Q have a unique common fixed point u^*;

(ii) There exists $\lambda \in (0, 1]$ such that the Jungck-Schaefer iteration $\{Qu_n\}_{n=0}^{\infty}$, defined by

$$Qu_{n+1} = (1 - \lambda)Qu_n + \lambda Pu_n, \quad n \geq 0 \tag{3.10}$$

converges to u^*, for any $u_0 \in B$.

Proof: *Since $c \geq 0$, we have two possible cases to consider (i.e $c = 0$ and $c > 0$).*

Case 1: *For $c = 0$, inequality (9) reduces to Definition (1.5) of the author in [6] and the prove follows the same argument of Theorem (2.2) in [6].*

Case 2: *When $c > 0$. Like the prove of theorem (3.1) above, we have*

$$\begin{aligned}
||Qu_n - Qu_{n+1}|| =&||P_\lambda u_{n-1} - P_\lambda u_n|| \\
\leq&\lambda\Big(\phi[||Qu_{n-1} - Qu_n||, \\
&||Qu_{n-1} - P_\lambda u_{n-1}||, ||Qu_n - P_\lambda u_n||, \\
&(||Qu_{n-1} - P_\lambda u_{n-1}||)^r \\
&(||Qu_n - P_\lambda u_{n-1}||)^p \\
&(||Qu_{n-1} - P_\lambda u_n||), \\
&||Qu_n - P_\lambda u_{n-1}|| \\
&(||Qu_{n-1} - P_\lambda u_{n-1}||)^m]\Big) \\
=&\lambda\Big(\phi[||Qu_{n-1} - Qu_n||, \\
&||Qu_{n-1} - Qu_n||, ||Qu_n - Qu_{n+1}||, \\
&(||Qu_{n-1} - Qu_n||)^r(||Qu_n - Qu_n||)^p \\
&(||Qu_{n-1} - Qu_{n+1}||), \\
&||Qu_n - Qu_n||(||Qu_{n-1} - Qu_n||)^m]\Big) \\
=&\lambda\Big(\phi[||Qu_{n-1} - Qu_n||, ||Qu_{n-1} - Qu_n||, \\
&||Qu_n - Qu_{n+1}||, 0, 0]\Big) \\
\leq&\psi(\lambda||Qu_{n-1} - Qu_n||),
\end{aligned}$$

$Since$ ψ is positively homogeneous function

i.e

$$||Qu_n - Qu_{n+1}|| \leq \psi(\lambda||Qu_{n-1} - Qu_n||). \tag{3.11}$$

Inductively from (3.11) we have

$$\begin{aligned}
||Qu_n - Qu_{n+1}|| &\leq \psi(\lambda||Qu_{n-1} - Qu_n||) \\
&\leq \psi^2(\lambda||Qu_{n-2} - Qu_{n-1}||) \\
&\leq \psi^3(\lambda||Qu_{n-3} - Qu_{n-2}||) \\
&\leq \psi^n(\lambda||Qu_0 - Qu_1||),
\end{aligned}$$

i.e

$$||Qu_n - Qu_{n+1}|| \leq \psi^n(\lambda||Qu_0 - Qu_1||) \tag{3.12}$$

By repeated application of triangle inequality on (3.12), for any $p \in \mathbb{N}$ we have:

$$\begin{aligned}
||Qu_n - Qu_{n+p}|| \leq& \sum_{k=n}^{n+p-1} \psi^k(\lambda||Qu_0 - Qu_1||) \\
=& \sum_{k=0}^{n+p-1} \psi^k(\lambda||Qu_0 - Qu_1||) \\
&- \sum_{k=0}^{n-1} \psi^k(\lambda||Qu_0 - Qu_1||).
\end{aligned} \tag{3.13}$$

Now, since ψ is a c-comparison function, it follows from (3.13) that $||Q_\lambda u_n - Q_\lambda u_{n+p}|| \to 0$ as $n \to \infty$.

Hence, $\{Q_\lambda u_n\}_{n=0}^{\infty}$ is a Cauchy sequence in Banach space B, then, there exists $u^* \in B$ such that

$$\lim_{n\to\infty} Qu_n = \lim_{n\to\infty} P_\lambda u_{n-1} = u^*.$$

With continuity of Q and commutativity of P and Q, we have the following:

$$Qu^* = Q(\lim_{n\to\infty} Qu_n) = \lim_{n\to\infty} Q^2u_n \tag{3.14}$$

$$Qu^* = Q(\lim_{n\to\infty} P_\lambda u_n) = \lim_{n\to\infty}(QP_\lambda u_n) = \lim_{n\to\infty}(P_\lambda Qu_n) \tag{3.15}$$

Thus, we have

$$\begin{aligned}||P_\lambda(Qu_n) - P_\lambda u^*|| \leq &\lambda\Big(\phi[||Q^2u_n - Qu^*||,\\ &||Q^2u_n - P_\lambda(Qu_n)||, ||Qu^* - P_\lambda u^*||,\\ &(||Q(Qu_n) - P_\lambda(Qu_n)||)^r\\ &(||Qu^* - P_\lambda(Qu_n)||)^p\\ &(||Q(Qu_n) - P_\lambda u^*||),\\ &||Qu^* - P_\lambda(Qu_n)||\\ &(||Q(Qu_n) - P_\lambda(Qu_n)||)^m]\Big).\\ =&\lambda\Big(\phi[||Q^2u_n - Qu^*||,\\ &||Q^2u_n - P_\lambda(Qu_n)||, ||Qu^* - P_\lambda u^*||,\\ &(||Q^2u_n - P_\lambda(Qu_n)||)^r\\ &(||Qu^* - P_\lambda(Qu_n)||)^p\\ &(||Q^2u_n - P_\lambda u^*||),\\ &||Qu^* - P_\lambda(Qu_n)||\\ &(||Q^2u_n - P_\lambda(Qu_n)||)^m]\Big).\end{aligned} \tag{3.16}$$

Applying (3.14) and (3.15) into (3.16), as $n \to \infty$ gives

$$\begin{aligned}||Qu^* - P_\lambda u^*|| \leq &\lambda\Big(\phi[||Qu^* - Qu^*||,\\ &||Qu^* - Qu^*||, ||Qu^* - P_\lambda u^*||,\\ &(||Qu^* - Qu^*)||)^r\\ &(||Qu^* - Qu^*)||)^p(||Qu^* - P_\lambda u^*||),\\ &||Qu^* - Qu^*||\\ &(||Qu^* - Qu^*)||)^m]\Big).\\ =&\lambda\Big(\phi[0, 0, ||Qu^* - P_\lambda u^*||, 0, 0]\Big)\\ \leq&\psi(\lambda\times 0) = \psi(0) = 0.\end{aligned}$$

Therefore, we have $Qu^* = P_\lambda u^*$.

And again, we also have

$$\begin{aligned}||P_\lambda u_n - P_\lambda u^*|| \leq &\lambda\Big(\phi[||Qu_n - Qu^*||,\\ &||Qu_n - P_\lambda u_n)||, ||Qu^* - P_\lambda u^*||,\\ &(||Qu_n - P_\lambda u_n)||)^r\\ &(||Qu^* - P_\lambda u_n)||)^p(||Qu_n - P_\lambda u^*||),\\ &||Qu^* - P_\lambda u_n||(||Qu_n - P_\lambda u_n||)^m]\Big).\end{aligned}$$

Taking limit as $n \to \infty$ gives

$$\begin{aligned}||u^* - P_\lambda u^*|| \leq &\lambda\Big(\phi[||u^* - Qu^*||,\\ &||u^* - u^*||, ||Qu^* - P_\lambda u^*||,\\ &(||u^* - u^*||)^r(||Qu^* - u^*||)^p\\ &(||u^* - P_\lambda u^*||), ||Qu^* - u^*||\\ &(||u^* - u^*)||)^m]\Big)\\ =&\lambda\Big(\phi[||u^* - P_\lambda u^*||, 0, 0, 0, 0]\Big)\\ \leq&\psi(\lambda\times 0) = \psi(0) = 0.\end{aligned}$$

This implies that, $u^* = P_\lambda u^*$. Hence, $Su^* = P_\lambda u^* = u^*$.

Now, we prove the uniqueness of this common fixed point. Suppose not, then there exists $u^* \in B$, such that $Qu^* = P_\lambda u^* = u^*, Qv^* = P_\lambda v^* = v^*$, we have

$$\begin{aligned}||u^* - v^*|| =&||P_\lambda u^* - P_\lambda v^*||\\ \leq&\lambda\Big(\phi[||Qu^* - Qv^*||,\\ &||Qu^* - P_\lambda u^*||, ||Qv^* - P_\lambda v^*||,\\ &(||Qu^* - P_\lambda u^*||)^r\\ &(||Qv^* - P_\lambda u^*||)^p(||Qu^* - P_\lambda v^*||),\\ &||Qv^* - P_\lambda u^*||\\ &(||Qu^* - P_\lambda u^*||,)^m]\Big)\\ =&\lambda\Big(\phi[||u^* - v^*||, 0, 0, 0, 0]\Big)\\ \leq&\psi(\lambda\times 0) = \psi(0) = 0.\end{aligned}$$

We conclude that, $u^* = v^*$

Corollary 3.4. *Given a Banach space $(B, ||.||)$ and let $P, Q : B \to B$ be commuting and an enriched Jungck-contraction. If Q is continuous and $P(B) \subseteq Q(B)$, then:*

(i) P_λ and Q have a unique common fixed point u^*;

(ii) There exists $\lambda \in (0, 1]$ such that the Jungck-Schaefer iteration $\{Qu_n\}_{n=0}^\infty$ converges to u^*, the unique common fixed point of P_λ and Q, for any $u_0 \in B$.

***Proof**: This follows the same line of argument of the prove of Theorem 3.1.*

Corollary 3.5. *Given a Banach space $(B, ||.||)$ and let $P, Q : B \to B$ be commuting and an enriched $\psi-$ Jungck-contraction. If Q is continuous and $P(B) \subseteq Q(B)$. Then:*

(i) P_λ and Q have a unique common fixed point u^*;

(ii) There exists $\lambda \in (0, 1]$ such that the Jungck-Schaefer iteration $\{Qu_n\}_{n=0}^\infty$ converges to u^*, the unique common fixed point of P_λ and Q, for any $u_0 \in B$.

***Proof**: This follows the same line of argument of the prove of Theorem 3.2.*

The below theorems established unique common fixed point of sequence of generalized enriched Jungck operators.

Theorem 3.6. *Given a Banach space $(B, ||.||)$, and S a continuous self map operator on B. If S commute with each $\{T_i\}_{i=1}^k : B \to B$ such that T_i is a sequence of generalized enriched Jungck contraction and $T_i(B) \subseteq S(B)$ $(for\ each\ i)$. Then:*

(i) All $(T_i)_\lambda$ and S have a unique common fixed point u^*; and

(ii) There exists $\lambda \in (0,1]$ such that the Jungck-Schaefer iteration $\{Su_n\}_{n=0}^{\infty}$, defined by

$$Su_{n+1} = (1-\lambda)Su_n + \lambda T_i u_n, \quad n \geq 0, \tag{3.17}$$

converges to u^*, for any $u_0 \in B$.

Proof: *Since $c \geq 0$, there are two possible cases to be considered (i.e $c = 0$ and $c > 0$).*

Case 1: *For $c = 0$, then for each i inequality (2.1) becomes*

$$\begin{aligned}||T_iu - T_iv|| \leq & \phi[||Su - Sv||, \\ & ||Su - T_iu||, ||Sv - T_iv||, \\ & (||Su - T_iu||)^r(||Sv - T_iu||)^p \\ & (||Su - T_iv||), ||Sv - T_iu|| \\ & (||Su - T_iu||)^m].\end{aligned}$$

Now, since $T_i(B) \subseteq S(B)$ $(for\ each\ i)$, and by Jungck iteration, $Su_1 = T_iu_0$ (for each i), taking any $u_0 \in B$, for each $i \in \mathbb{N}$, we have

$$\begin{aligned}||T_1u_0 - T_1u_1|| \leq & \phi[||Su_0 - Su_1||, \\ & ||Su_0 - T_1u_0||, ||Su_1 - T_1u_1||, \\ & ||Su_0 - T_1u_0||)^r \\ & (||Su_1 - T_1u_0||)^p(||Su_0 - T_1u_1||), \\ & ||Su_1 - T_1u_0||(||Su_0 - T_1u_0||)^m] \\ = & \phi[||Su_0 - Su_1||, ||Su_0 - Su_1||, \\ & ||Su_1 - Su_2||, (||Su_0 - Su_1||)^r \\ & (||Su_1 - Su_1||)^p(||Su_0 - Su_2||), \\ & ||Su_1 - Su_1||(||Su_0 - Su_1||)^m],\end{aligned}$$

i.e

$$\begin{aligned}||Su_1 - Su_2|| \leq & \phi[||Su_0 - Su_1||, \\ & ||Su_0 - Su_1||, \\ & ||Su_1 - Su_2||, 0, 0] \\ \leq & k.||Su_0 - Su_1||\end{aligned}$$

Also,

$$\begin{aligned}||Su_2 - Su_3|| \leq & \phi[||Su_1 - Su_2||, \\ & ||Su_1 - Su_2||, \\ & ||Su_2 - Su_3||, 0, 0] \\ \leq & k.||Su_1 - Su_2|| \\ = & k^2.||Su_0 - Su_1||,\end{aligned}$$

continue this way, we have

$$\begin{aligned}||T_iu_{n-1} - T_iu_n|| = & ||Su_n - Su_{n+1}|| \\ \leq & \phi[||Su_{n-1} - Su_n||, \\ & ||Su_{n-1} - Su_n||, \\ & ||Su_n - Su_{n+1}||, 0, 0] \\ \leq & k^n.||Su_0 - Su_1||.\end{aligned}$$

That is

$$\begin{aligned}||Su_n - Su_{n+1}|| \leq & k^n.||Su_0 - Su_1|| \to 0, \\ & as\ n \to \infty.\end{aligned}$$

Hence, $\{Su_n\}_{n=0}^{\infty}$ is a Cauchy sequence in B, then, there exists $u^* \in B$ such that for each i

$$\lim_{n\to\infty} Su_n = \lim_{n\to\infty} T_iu_{n-1} = u^*.$$

With continuity of S and its commutativity with each T_i, we have the following:

$$Su^* = S(\lim_{n\to\infty} Su_n) = \lim_{n\to\infty} S^2u_n \tag{3.18}$$

$$Su^* = S(\lim_{n\to\infty} T_iu_n) = \lim_{n\to\infty}(ST_iu_n) = \lim_{n\to\infty}(T_iSu_n) \tag{3.19}$$

Thus, using our contraction condition again with $u = Su_n$, $v = u^*$, we have, for each T_i

$$\begin{aligned}||T_i(Su_n) - T_iu^*|| \leq & \phi[||(S(Su_n) \\ & - Su^*||, ||S(Su_n) - T_i(Su_n)||, \\ & ||Su^* - T_iu^*||, \\ & ||S(Sx_n) - T_i(Su_n)||^r \\ & (||Su^* - T_i(Su_n)||)^p \\ & ||S(Su_n) - T_iu^*||, \\ & ||Su^* - T_i(Su_n)|| \\ & (||S(Su_n) - T_i(Su_n)||)^m].\end{aligned}$$

Using the continuity of S and taking limits in the above together with the application of (3.18) and (3.19) yield,

$$\begin{aligned}||S^2u_n - T_iu^*|| \leq & \phi[||S^2u_n - Su^*||, \\ & ||S^2u_n - T_i(Su_n)||, ||Su^* - T_iu^*||, \\ & (||S^2u_n - T_i(Su_n)||)^r \\ & ||Su^* - T_i(Su_n)||^p \\ & ||S^2u_n - T_iu^*||, \\ & ||Su^* - T_i(Su_n)|| \\ & (||S^2u_n - T_i(Su_n))^m],\end{aligned}$$

as $n \to \infty$ we have,

$$\begin{aligned}||Su^* - T_iu^*|| \leq & \phi[||Su^* - Su^*||, \\ & ||Su^* - Su^*||, ||Su^* - T_iu^*||, \\ & (||Su^* - Su^*||)^r \\ & ||Su^* - Su^*||^p||Su^* - (T_i)_\lambda u^*||, \\ & ||Su^* - Su^*||(||Su^* - Su^*||)^m] \\ = & \phi(0, 0, ||Sx^* - T_iu^*||, 0, 0) \\ \leq & k^n.0 = 0.\end{aligned}$$

Hence, $Su^* = T_iu^*$. And this implies that $Su^* = u^* = T_iu^*$.

Now, for the uniqueness of the fixed point. Suppose not, then there exists $u^* \in B$ such that $T_iu^* = Su^* = u^*$, and $T_iv^* = Sv^* = v^*$, and we have,

$$\begin{aligned}||u^* - v^*|| = & ||T_iu^* - T_iv^*|| \\ \leq & \phi||Su^* - Sv^*||, \\ & ||Su^* - T_iu^*||, ||Sv^* - T_iv^*||, \\ & (||Su^* - T_iu^*||)^r \\ & ||Sv^* - T_iu^*||^p||Su^* - T_iv^*||, \\ & ||Sv^* - T_iu^*||(||Su^* - T_iu^*)^m],\end{aligned}$$

so,

$$\begin{aligned}||u^* - v^*|| \leq &\phi[||u^* - v^*||,\\ &||u^* - u^*||, ||v^* - v^*||,\\ &(||u^* - u^*||)^r(||v^* - u^*||)^p\\ &||u^* - v^*||,\\ &||v^* - u^*||(||u^* - u^*||)^m]\\ =&\phi(||u^*, v^*||, 0, 0, 0, 0)\\ \leq &k.0 = 0.\end{aligned}$$

Hence $u^* = v^*$.

Case2: When $c > 0$. Considering iteration defined by (3.17) and for $\lambda = \frac{1}{c+1}$, then we have that

$$c = \frac{1-\lambda}{\lambda}, \tag{3.20}$$

hence, an enriched generalized Akram-Jungck contraction becomes

$$\begin{aligned}||\frac{1-\lambda}{\lambda}(Su - Sv)\\ +T_iu - T_iv|| \quad \leq \quad &\lambda\Big(\phi[||Su - Sv||,\\ &||Su - (T_i)_\lambda u||,\\ &||Sv - (T_i)_\lambda v||, (||Su - (T_i)_\lambda u||)^r\\ &(||Sv - T_iu||)^p(||Su - T_iv||),\\ &||Sv - T_iu||(||Su - T_iu||)^m]\Big).\end{aligned}$$

We have that

$$\begin{aligned}||(T_i)_\lambda u - (T_i)_\lambda v|| \leq &\lambda\Big(\phi[||S_\lambda u - S_\lambda v||,\\ &||S_\lambda u - (T_i)_\lambda u||, ||S_\lambda v - (T_i)_\lambda v||,\\ &(||S_\lambda u - (T_i)_\lambda u||)^r\\ &(||S_\lambda v - (T_i)_\lambda u||)^p(||S_\lambda u - (T_i)_\lambda v||),\\ &||S_\lambda v - (T_i)_\lambda u||\\ &(||S_\lambda u - (T_i)_\lambda u||)^m]\Big).\end{aligned} \tag{3.21}$$

Now, considering Jungck-Schaefer iterative process $\{Su_n\}_{n=0}^{\infty}$, which actually coincides with Jungck iteration associated with T_λ i.e

$$Su_n = (T_i)_\lambda u_{n-1}.$$

Let $u = u_n$ and $v = u_{n+1}$, so

$$||Su_n - Su_{n+1}|| = ||(T_i)_\lambda u_{n-1} - (T_i)_\lambda u_n|| \quad \text{(for each } i\text{)},$$

that is

$$\begin{aligned}||Su_1 - Su_2|| =&||(T_1)_\lambda u_0 - (T_1)_\lambda u_1||\\ \leq&\lambda\Big(\phi[||Su_0 - Su_1||,\\ &||Su_0 - (T_1)_\lambda u_0||, ||Su_1 - (T_1)_\lambda u_1||,\\ &(||Su_0 - (T_1)_\lambda u_0||)^r\\ &(||Su_1 - (T_1)_\lambda u_0||)^p\\ &(||Su_0 - (T_1)_\lambda u_1||),\\ &||Su_1 - (T_1)_\lambda u_0||\\ &(||Su_0 - (T_1)_\lambda u_0||)^m]\Big)\\ =&\lambda\Big(\phi[||Su_0 - Su_1||,\\ &||Su_0 - Su_1||, ||Su_1 - Su_2||,\\ &(||Su_0 - Su_1||)^r(||Su_1 - Su_1||)^p\\ &(||Su_0 - Su_2||),\\ &||Su_1 - Su_1||(||Su_0 - Su_1||)^m]\Big)\\ =&\lambda\Big(\phi[||Su_0 - Su_1||, ||Su_0 - Su_1||,\\ &||Su_1 - Su_2||, 0, 0]\Big)\\ \leq&\mu(||Su_0 - Su_1||)\end{aligned}$$

where $\mu = \lambda \times k$,that is

$$||Su_1 - Su_2|| \leq \mu(||Su_0 - Su_1||). \tag{3.22}$$

Inductively from (3.22) we have

$$\begin{aligned}||Su_n - Su_{n+1}|| \leq&\mu(||Su_{n-1} - Su_n||)\\ \leq&\mu^2(||Su_{n-2} - Su_{n-1}||)\\ \leq&\mu^3(||Su_{n-3} - Su_{n-2}||)\\ \leq&\mu^n(||Su_0 - Su_1||).\end{aligned}$$

i.e

$$||Su_n - Su_{n+1}|| \leq \mu^n(||Su_0 - Su_1||). \tag{3.23}$$

By repeated application of triangle inequality on (3.23), for any $p \in \mathbb{N}$ we have:

$$\begin{aligned}||Su_n - Su_{n+p}|| \leq&\frac{\mu^n(1-\mu^p)}{1-\mu}(||Su_0 - Su_1||)\\ \to&0, \; as \; n \to \infty\end{aligned} \tag{3.24}$$

Hence, $\{Su_n\}_{n=0}^{\infty}$ is a Cauchy sequence in Banach space B, then, there exists $u^* \in B$ such that

$$\lim_{n\to\infty} S_\lambda u_n = \lim_{n\to\infty}(T_i)_\lambda u_{n-1} = u^*. \; (for \; each \; i)$$

With continuity of S and commutativity of each $(T_i)_\lambda$ and S, we have the following:

$$Su^* = S(\lim_{n\to\infty} Su_n) = \lim_{n\to\infty} S^2 u_n, \tag{3.25}$$

$$\begin{aligned}Su^* \quad &= \quad S(\lim_{n\to\infty}(T_i)_\lambda u_n)\\ &= \quad \lim_{n\to\infty}(S(T_i)_\lambda u_n)\\ &= \quad \lim_{n\to\infty}((T_i)_\lambda Su_n).\end{aligned} \tag{3.26}$$

Thus, using inequality (3.21), with $u = Su_n$ and $v = u^*$, we have

$$\begin{aligned}||(T_i)_\lambda(Su_n) - (T_i)_\lambda u^*|| \leq & \lambda\Big(\phi[||S^2u_n - Su^*||, \\ & ||S^2u_n - (T_i)_\lambda(Su_n)||, \\ & ||Su^* - (T_i)_\lambda u^*||, \\ & (||S(Su_n) - (T_i)_\lambda(Su_n)||)^r \\ & (||Su^* - (T_i)_\lambda(Su_n)||)^p \\ & (||S(Su_n) - (T_i)_\lambda u^*||), \\ & ||Su^* - (T_i)_\lambda(Su_n)|| \\ & (||S(Su_n) - (T_i)_\lambda(Su_n)||)^m]\Big) \\ = & \lambda\Big(\phi[||S^2u_n - Su^*||, \\ & ||S^2u_n - (T_i)_\lambda(Su_n)||, \\ & ||Su^* - (T_i)_\lambda u^*||, \\ & (||S^2u_n - (T_i)_\lambda(Su_n)||)^r \\ & (||Su^* - (T_i)_\lambda(Su_n)||)^p \\ & (||S^2u_n - (T_i)_\lambda u^*||), \\ & ||Su^* - (T_i)_\lambda(Su_n)|| \\ & (||S^2u_n - (T_i)_\lambda(Su_n)||)^m]\Big).\end{aligned} \tag{3.27}$$

Applying (3.25) and (3.26) into (3.27), as $n \to \infty$ gives

$$\begin{aligned}||Su^* - (T_i)_\lambda u^*|| \leq & \lambda\Big(\phi[||Su^* - Su^*||, \\ & ||Su^* - Su^*||, ||Su^* - (T_i)_\lambda u^*||, \\ & (||Su^* - Su^*)||)^r(||Su^* - Su^*)||)^p \\ & (||Su^* - (T_i)_\lambda u^*||), \\ & ||Su^* - Su^*||(||Su^* - Su^*)||)^m]\Big) \\ = & \lambda\Big(\phi[0, 0, ||S_\lambda u^* - (T_i)_\lambda u^*||, 0, 0]\Big) \\ \leq & \mu(0) = 0.\end{aligned}$$

Therefore, we have $S_\lambda u^* = (T_i)_\lambda u^*$.

And again by inequality (3.21), we also have

$$\begin{aligned}||(T_i)_\lambda u_n - (T_i)_\lambda u^*|| \leq & \lambda\Big(\phi[||Su_n - Su^*||, \\ & ||Su_n - (T_i)_\lambda u_n)||, \\ & ||Su^* - (T_i)_\lambda u^*||, \\ & (||Su_n - (T_i)_\lambda u_n)||)^r \\ & (||Su^* - (T_i)_\lambda u_n)||)^p \\ & (||Su_n - (T_i)_\lambda u^*||), \\ & ||Su^* - (T_i)_\lambda u_n|| \\ & (||Su_n - (T_i)_\lambda u_n||)^m]\Big).\end{aligned}$$

Taking limit as $n \to \infty$ gives

$$\begin{aligned}||u^* - (T_i)_\lambda u^*|| \leq & \lambda\Big(\phi[||u^* - Su^*||, ||u^* - u^*||, \\ & ||Su^* - (T_i)_\lambda u^*||, (||u^* - u^*||)^r \\ & (||Su^* - u^*||)^p \\ & (||u^* - (T_i)_\lambda u^*||), ||Su^* - u^*|| \\ & (||u^* - u^*)||)^m]\Big) \\ = & \lambda\Big(\phi[||u^* - (T_i)_\lambda u^*||, 0, 0, 0, 0]\Big) \\ \leq & \mu(0) = 0.\end{aligned}$$

This implies that, $u^* = (T_i)_\lambda u^*$.

Hence,

$$Su^* = (T_i)_\lambda u^* = u^*.$$

Now, we prove the uniqueness of this common fixed point.

Suppose not, then there exists $u^* \in B$, such that $Su^* = (T_i)_\lambda u^* = u^*$, $Sv^* = (T_i)_\lambda v^* = v^*$, we have the following:

$$\begin{aligned}||u^* - v^*|| = & ||(T_i)_\lambda u^* - (T_i)_\lambda v^*|| \\ \leq & \lambda\Big(\phi[||Su^* - Sv^*||, \\ & ||Su^* - (T_i)_\lambda u^*||, \\ & ||Sv^* - (T_i)_\lambda v^*||, \\ & (||Su^* - (T_i)_\lambda u^*||)^r \\ & (||Sv^* - (T_i)_\lambda u^*||)^p \\ & (||Su^* - (T_i)_\lambda v^*||), \\ & ||Sv^* - (T_i)_\lambda u^*|| \\ & (||Su^* - (T_i)_\lambda u^*||,)^m]\Big) \\ = & \lambda\Big(\phi[||u^* - v^*||, 0, 0, 0, 0]\Big) \\ \leq & \mu(0) = 0.\end{aligned}$$

Hence, $u^* = v^*$

Theorem 3.7. *Given a Banach space $(B, ||.||)$ and Q a continuous self map operator on B. If Q commute with each $\{P_i\}_{i=1}^k : B \to B$ such that P_i is a sequence of generalized enriched ψ-Jungck contraction and $P_i(B) \subseteq Q(B)$ $(for\ each\ i)$. Then:*

(i) All $(P_i)_\lambda$ and S have a unique common fixed point u^*; and

(ii) There exists $\lambda \in (0, 1]$ such that the Jungck-Schaefer iteration $\{Qu_n\}_{n=0}^\infty$, defined by

$$Qu_{n+1} = (1 - \lambda)Qu_n + \lambda Pu_n, \quad n \geq 0 \tag{3.28}$$

converges to u^*, for any $u_0 \in B$.

Proof: *Just like the prove of theorem 3.5, we have two possible cases to consider (i.e $c = 0$ and $c > 0$).*

Case1: *For $c = 0$: Taking any $u_0 \in B$, for each $i \in \mathbb{N}$, we have*

$$\begin{aligned}||P_iu_{n-1} - P_iu_n|| = & ||Qu_n - Qu_{n+1}|| \\ \leq & \phi[||Qu_{n-1} - Qu_n||, \\ & ||Qu_{n-1} - Qu_n||, \\ & ||Qu_n - Qu_{n+1}||, 0, 0] \\ \leq & \psi^n(||Qu_0 - Qu_1||).\end{aligned}$$

And using triangle inequality inductively, together with the properties of ψ, we say $\{Qu_n\}_{n=0}^{\infty}$ is a Cauchy sequence in B, then, there exists $u^* \in B$ such that for each i

$$\lim_{n\to\infty} Qu_n = \lim_{n\to\infty} P_i u_{n-1} = u^*.$$

And we continue just as in the prove of theorem 3.5 above.

Case2: $(c > 0)$ We have that

$$\begin{aligned} ||Qu_n - Qu_{n+1}|| &= ||(P_i)_\lambda u_{n-1} - (P_i)_\lambda u_n|| \\ &\leq \psi(\lambda||Qu_{n-1} - Qu_n||) \\ &\leq \psi^2(\lambda||Qu_{n-2} - Qu_{n-1}||) \\ &\leq \psi^3(\lambda||Qu_{n-3} - Qu_{n-2}||) \\ &\leq \psi^n(\lambda||Qu_0 - Qu_1||). \end{aligned}$$

i.e.,

$$||Qu_n - Qu_{n+1}|| \leq \psi^n(\lambda||Qu_0 - Qu_1||) \tag{3.29}$$

Using triangle inequality inductively, together with the properties of ψ, we say

$\{Qu_n\}_{n=0}^{\infty}$ is a Cauchy sequence in Banach space B, then, there exists $u^* \in B$ such that

$$\lim_{n\to\infty} Q_\lambda u_n = \lim_{n\to\infty} (P_i)_\lambda u_{n-1} = u^*. \, (for\ each\ i)$$

The rest of the prove follows the same argument as that of theorem 3.5 above.

Corollary 3.8. *Let $(B, ||.||)$ be a Banach space and let S be a continuous self map operator on B. If S commute with each $\{T_i\}_{i=1}^k : B \to B$ such that T_i is a sequence of enriched Jungck-contraction and $T_i(B) \subseteq S(B)$ $(for\ each\ i)$. Then:*

(i) All $(T_i)_\lambda$ and S have a unique common fixed point u^*; and

(ii) There exists $\lambda \in (0, 1]$ such that the Jungck-Schaefer iteration $\{Su_n\}_{n=0}^{\infty}$, defined by

$$Su_{n+1} = (1 - \lambda)Su_n + \lambda Tu_n, \quad n \geq 0 \tag{3.30}$$

converges to u^*, for any $u_0 \in B$.

Proof: This follows the same line of argument of the prove of theorem 3.5.

Corollary 3.9. *$(B, ||.||)$ be a Banach space and let S be a continuous self map operator on B. If S commute with each $\{T_i\}_{i=1}^k : B \to B$ such that T_i is a sequence of enriched ψ-Jungck contraction and $T_i(B) \subseteq S(B)$ $(for\ each\ i)$. Then:*

(i) All $(T_i)_\lambda$ and S have a unique common fixed point u^*; and

(ii) There exists $\lambda \in (0, 1]$ such that the Jungck-Schaefer iteration $\{Su_n\}_{n=0}^{\infty}$, defined by

$$Su_{n+1} = (1 - \lambda)Su_n + \lambda Tu_n, \quad n \geq 0 \tag{3.31}$$

converges to u^*, for any $u_0 \in B$.

Proof: *This follows the same line of argument of the prove of Theorem 3.7.*

4. Approximating common fixed point of compatible enriched Jungck-generalized contractive mappings

The remaining part of this paper focuses on the weaker form of commuting maps for the existence and uniqueness of common fixed point of the above discussed generalizations of enriched Jungck contractions.

Theorem 4.1. *Let $(B, ||.||)$ be a Banach space and let S be a continuous self mapping on B. If S is compatible with each $\{T_i\}_{i=1}^k : B \to B$ such that T_i is a sequence of generalized enriched Jungck contraction (i.e Definition 2.3) and $T(B) \subseteq S(B)$, Then:*

- All $(T_i)_\lambda$ and S have a unique common fixed point w; and
- there exists $\lambda \in (0, 1]$ such that the Jungck-Schaefer iteration $\{Su_n\}_{n=0}^{\infty}$, defined by

$$\begin{aligned} Su_{n+1} &= (T_i)_\lambda u_n \\ &= (1 - \lambda)Su_n + \lambda T_i u_n, \ \ n \geq 0, \end{aligned} \tag{4.1}$$

converges to w

Proof. The prove of second part is the same argument as that of theorem 3.6 above. That is,

$$\lim_{n\to\infty} Su_{n+1} = \lim_{n\to\infty} (T_i)_\lambda u_n = w, \tag{4.2}$$

and by continuity of S, we have from (4.2)

$$\lim_{n\to\infty} S(Su_n) = Sw,$$

Now, since S and T_i are compatible for each i and

$$Su_{n+1} = (T_i)_\lambda u_n = (1 - \lambda)Su_n + \lambda T_i u_n,$$

then, for each i, $(T_i)_\lambda$ and S are also compatible. Therefore, by Lemma 2.12, we have

$$\lim_{n\to\infty} (T_i)_\lambda Su_n = Sw.$$

From inequality (3.21), with $u = Su_n$ and $v = u_n$, we have

$$\begin{aligned} ||(T_i)_\lambda Su_n - (T_i)_\lambda u_n|| \leq &\lambda\Big(\phi[||S(Su_n) - Su_n||, \\ &||S(Su_n) - (T_i)_\lambda Su_n||, \\ &||Su_n - (T_i)_\lambda u_n||, \\ &(||S(Su_n) - (T_i)_\lambda Su_n||)^r \\ &(||Su_n - (T_i)_\lambda Su_n||)^p \\ &(||S(Su_n) - (T_i)_\lambda u_n||), \\ &||Su_n - (T_i)_\lambda Su_n|| \\ &(||S(Su_n) - (T_i)_\lambda Su_n||)^m]\Big). \end{aligned}$$

as $n \to \infty$ we have

$$\begin{aligned} ||Sw - w|| \leq &\lambda\Big(\phi[||Sw - w||, ||Sw - Sw||, \\ &||w - w||, (||Sw - Sw||)^r \\ &(||w - Sw||)^p \\ &(||Sw - w||), ||w - Sw|| \\ &(||Sw - Sw||)^m]\Big) \\ = &\lambda\Big(\phi[||Sw - w||, 0, 0, 0, 0)^m]\Big) \\ = &\mu(0) = 0. \end{aligned}$$

$\implies Sw = w.$

Again by Lemma 2.12 (b), together with the fact that S is continuous and

$T(B) \subseteq S(B), \implies (T_i)_\lambda$ is also continuous. Then, we have

$$(T_i)_\lambda w = Sw = w,$$

and

$$\lim_{n\to\infty} S(T_i)_\lambda u_n = (T_i)_\lambda w.$$

Now, we prove the uniqueness of the common fixed point.

Suppose not, then there exists $w^* \in B$, such that

$Sw^* = (T_i)_\lambda w^* = w^*$, $Sv^* = (T_i)_\lambda v^* = v^*$, we have the following:

$$\begin{aligned}
||w^* - v^*|| &= ||(T_i)_\lambda w^* - (T_i)_\lambda v^*|| \\
&\leq \lambda\Big(\phi[||Sw^* - Sv^*||, \\
&\quad ||Sw^* - (T_i)_\lambda w^*||, ||Sv^* - (T_i)_\lambda v^*||, \\
&\quad (||Sw^* - (T_i)_\lambda w^*||)^r \\
&\quad (||Sv^* - (T_i)_\lambda w^*||)^p \\
&\quad (||Sw^* - (T_i)_\lambda v^*||), \\
&\quad ||Sv^* - (T_i)_\lambda w^*|| \\
&\quad (||Sw^* - (T_i)_\lambda w^*||,)^m]\Big) \\
&= \lambda\Big(\phi[||w^* - v^*||, 0, 0, 0, 0]\Big) \\
&\leq \mu(0) = 0.
\end{aligned}$$

Hence, $w^* = v^*$ □

5. Conclusion

We have proved the existence and uniqueness of common fixed points of enriched-Jungck contractions, a generalization of enriched contractive definition of Berinde and Pacurar [7] in line with the result due to Akram [5] and Olatinwo and Omidire [6] in a Banach space setting. It is worth noting that our results have reinforced the convergence of Jungck-Schaefer iterative procedure to the unique common fixed points of more general class of enriched contraction definitions involving pair of commuting and compatible mappings. We extended the results in [7] to pair of commuting and compatible operators called, *generalized enriched Jungck-contractions*. We proved corresponding fixed point theorems for these type of operators and for sequences of generalized enriched Jungck-contractive pair of compatible operators. Our results unify, generalize and extend results in [4, 5, 6, 7, 16], and many other related results in literature.

References

[1] Banach S, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundamenta Mathematicae*, 3(1) (1922) 133–181. ISSN 1730-6329. https://doi.org/10.4064/fm-3-1-133-181.

[2] Rakotch E, A note on contractive mappings, *Proceedings of the American Mathematical Society*, 13(3) (1962) 459–465. ISSN 1088-6826. https://doi.org/10.1090/s0002-9939-1962-0148046-1.

[3] Kannan R, Some results on fixed points—II, *The American Mathematical Monthly*, 76(4) (1969) 405. ISSN 0002-9890. https://doi.org/10.2307/2316437.

[4] Jungck G, Commuting mappings and fixed points, *The American Mathematical Monthly*, 83(4) (1976) 261–263. ISSN 1930-0972. https://doi.org/10.1080/00029890.1976.11994093.

[5] Akram M, Zafar A & Siddiqui A, A general class of contractions: A-contractions, *Novi Sad J. Math*, 38(1) (2008) 25–33.

[6] Olatinwo M & Omidire O. Fixed point theorems of Akram-Banch type. In: *Nonlinear Anal. Forum*, vol. 21 (2016), pp. 55–64.

[7] Berinde V & Păcurar M, Approximating fixed points of enriched contractions in Banach spaces, *Journal of Fixed Point Theory and Applications*, 22(2) (2020) 1–10. ISSN 1661-7746. https://doi.org/10.1007/s11784-020-0769-9.

[8] Olatinwo M & Omidire O, Convergence of Jungck-Schaefer and Jungck-Kirk-Mann iterations to the unique common fixed point of Jungck generalized pseudo-contractive and Lipschitzian type mappings, *Journal of Advanced Math. Stud*, 14(1) (2021) 153–167.

[9] Omidire O J, Common fixed point theorems of some certain generalized contractive conditions in convex metric space settings, *International Journal of Mathematical Sciences and Optimization: Theory and Applications*, 10(3) (2024) 1–9. https://doi.org/10.6084/zenodo.13152753. URL http://ijmso.unilag.edu.ng/article/view/2172.

[10] Olatinwo M O, Convergence results for Jungck-type iterative processes in convex metric spaces, *Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica*, 51(1) (2012) 79–87.

[11] Olatinwo M & Omidire O, Some new convergence and stability results for Jungck generalized pseudo-contractive and Lipschitzian type operators using hybrid iterative techniques in the Hilbert space, *Rendiconti del Circolo Matematico di Palermo Series 2*, 72(2) (2023) 1067–1086.

[12] Singh S L, Bhatnagar C & Mishra S N, Stability of Jungck-type iterative procedures, *International Journal of Mathematics and Mathematical Sciences*, 2005(19) (2005) 3035–3043. ISSN 1687-0425. https://doi.org/10.1155/ijmms.2005.3035.

[13] Singh S, Hematulin A & Pant R, New coincidence and common fixed point theorems, *Applied General Topology*, 10(1) (2009) 121–130. ISSN 1576-9402. https://doi.org/10.4995/agt.2009.1792.

[14] Olatinwo M & Omidire O, Convergence results for Kirk-Ishikawa and some other iterative algorithms in arbitrary Banach space setting, *Transactions on Mathematical Programming and Applications*, 8(1) (2020) 01–11.

[15] Berinde V. *Iterative approximation of fixed points*. Springer Berlin Heidelberg (2007). ISBN 9783540722335. https://doi.org/10.1007/978-3-540-72234-2.

[16] Marchiş A, Common fixed point theorems for enriched Jungck contractions in Banach spaces, *Journal of Fixed Point Theory and Applications*, 23. https://doi.org/10.1007/s11784-021-00911-y.

[17] Jungck G, Compatible mappings and common fixed points, *International Journal of Mathematics and Mathematical Sciences*, 9(4) (1986) 771–779. ISSN 1687-0425. https://doi.org/10.1155/s0161171286000935.