Kathmandu University
Journal of Science, Engineering and Technology

López-Bonilla *et. al.*, Vol. 13, No. I, October 2017, pp 95-97.

SHORT COMMUNICATION OPEN ACCESS

AN IDENTITY FOR STIRLING NUMBERS OF THE SECOND KIND

[1]B. M. Tuladhar, [2]J. López-Bonilla*, [2]O. Salas-Torres
[1] Kathmandu University, Dhulikhel, Kavre, Nepal
[2] ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México

*Corresponding author's email: jlopezb@ipn.mx
Received 30 August, 2017; Revised 30 November, 2017

ABSTRACT
We obtain an identity satisfied by the Stirling numbers of the second kind.

Keywords: Stirling numbers, Binomial coefficients.

INTRODUCTION

Here we show the identity:

$$\sum_{j=m}^{n} (-1)^j \, j! \, S_n^{[j]} = (-1)^{m+n-1} m! \sum_{k=m-1}^{n-1} (-1)^k S_k^{[m-1]}, \qquad 1 \le m \le n, \tag{1}$$

where $S_r^{[k]}$ are the Stirling numbers of the second kind [1, 2]. The case $m = 0$ is very known [1]:

$$\sum_{j=0}^{n} (-1)^j \, j! \, S_n^{[j]} = (-1)^n, \tag{2}$$

because it is consequence of [1, 3]:

$$\sum_{j=0}^{n} \binom{x}{j} j! \, S_n^{[k]} = x^n, \tag{3}$$

for $x = -1$.

An identity involving $S_r^{[k]}$

The Stirling numbers of the second kind are given by the Euler's expression [1, 3]:

$$S_n^{[j]} = \frac{(-1)^j}{j!} \sum_{q=0}^{j} (-1)^q \binom{j}{q} q^n, \tag{4}$$

therefore:

Kathmandu University
Journal of Science, Engineering and Technology

López-Bonilla *et. al.*, Vol. 13, No. I, October 2017, pp 95-97.

$$\sum_{j=m}^{n}(-1)^j\, j!\, S_n^{[j]} = \left(\sum_{j=0}^{n} - \sum_{j=0}^{m-1}\right)\sum_{q=0}^{j}(-1)^q\binom{j}{q}q^n,$$

$$= \sum_{q=0}^{n}(-1)^q\, q^n \sum_{j=q}^{n}\binom{j}{q} - \sum_{q=0}^{m-1}(-1)^q\, q^n\sum_{j=q}^{m-1}\binom{j}{q},$$

$$= (n+1)\sum_{q=0}^{n}(-1)^q\binom{n}{q}\frac{q^n}{q+1} - m\sum_{q=0}^{m-1}(-1)^q\binom{m-1}{q}\frac{q^n}{q+1}, \qquad (5)$$

where was applied the property [4, 5]:

$$\sum_{j=q}^{N}\binom{j}{q} = \binom{N+1}{q+1} = \frac{N+1}{q+1}\binom{N}{q}. \qquad (6)$$

On the other hand, from [1] we have the relation:

$$\sum_{q=0}^{N}(-1)^q\binom{N}{q}\frac{q^n}{q+x} = (-1)^n x^{n-1}\left[\frac{1}{\binom{N+x}{N}} - (-1)^N N!\sum_{k=0}^{n-1}\frac{(-1)^k}{x^k}S_k^{[N]}\right], \qquad x \neq 0, \qquad (7)$$

which for $x = 1$ implies:

$$\sum_{q=0}^{N}(-1)^q\binom{N}{q}\frac{q^n}{q+1} = \frac{(-1)^n}{N+1} - (-1)^{N+n} N!\sum_{k=0}^{n-1}(-1)^k S_k^{[N]}, \qquad (8)$$

thus:

$$\sum_{q=0}^{N}(-1)^q\binom{N}{q}\frac{q^n}{q+1} = \frac{(-1)^n}{N+1}, \qquad n \le N, \qquad (9)$$

because, $S_k^{[N]} = 0$ for $k < N$. Hence from (8) and (9):

$$\sum_{q=0}^{N}(-1)^q\binom{N}{q}\frac{q^n}{q+1} = \begin{cases} \dfrac{(-1)^n}{n+1}, & N = n \ge 0, \\[2ex] \dfrac{(-1)^n}{m} + (-1)^{m+n}(m-1)!\sum_{k=m-1}^{n-1}(-1)^k S_k^{[m-1]}, & 0 \le N = m-1 \le n-1, \end{cases} \qquad (10)$$

whose application into (5) gives the identity (1), Q.E.D.

López-Bonilla *et. al.*, Vol. 13, No. I, October 2017, pp 95-97.

REFERENCES

[1] Quaintance J & Gould H W, *Combinatorial identities for Stirling numbers*, World Scientific, Singapore, 2016.

[2] Barrera-Figueroa V, López-Bonilla J & López-Vázquez R, On Stirling numbers of the second kind, *Prespacetime Journal*, 8 (5) (2017), 572-574

[3] Arakawa T, Ibukiyama T & Kaneko M, Bernoulli numbers and zeta functions, *Springer*, Japan 2014.

[4] Gould H W, *Combinatorial identities*, Morgantown, W. Va., 1972.

[5] Graham R L, Knuth D E & Patashnik O, Concrete Mathematics: a foundation for computer science, *Addison-Wesley*, Amsterdam, 1994.