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**EXISTENCE OF MILD AND CLASSICAL SOLUTIONS OF
NONLINEAR INTEGRODIFFERENTIAL EQUATION IN BANACH
SPACE**

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ABSTRACT

In this paper, we use the theory of analytic semigroups to prove the existence and uniqueness of local mild and then classical solution of a class of nonlinear integrodifferential equations in Banach space.

KEYWORDS: Parabolic equation, Analytic semigroup, Mild and classical solutions.

MSC 2010 Codes: 34G20, 34K30, 35K55, 35K90.

INTRODUCTION

The aim of this paper is to study the following integrodifferential equation in Banach space X :

$$\frac{du(t)}{dt} + Au(t) = f \left(t, u(t), \int_{t_0}^t k(t,s,u(s))ds, \int_{t_0}^t h(t,s,u(s))ds \right) + \int_{t_0}^t B(t-s)g(s,u(s))ds, t > t_0. \quad (1.1)$$

$$u(t_0) = u_0 \quad (1.2)$$

We assume that $-A$ generates an analytic semi-group, $T(t), t \geq 0$ on X , the function B is real valued and locally integrable $[0, \infty)$. Let $f : [0, \infty) \times X \times X \times X \rightarrow X$, $g : [0, \infty) \times X \rightarrow X$ and $k, h : [0, \infty) \times [0, \infty) \times X \rightarrow X$ be given nonlinear operator. Our aim is to prove the existence and uniqueness of mild solutions to (1.1) – (1.2). We also establish the regularity of mild solution to (1.1) – (1.2). Further, under an additional condition of Hölder continuity on B , we prove that the mild solution to (1.1) – (1.2) is in fact a classical solution.

Equations (1.1) – (1.2) represent an abstract formulation of certain classes of parabolic integrodifferential equations. This type of equation models the physical phenomena involving certain type of memory effects. Several authors have studied the existence of solutions of the same type of problems by using different technique, we refer to [1, 3-8, 10, 11, 13].

The results obtained in this paper are generalization of the results given by Bahuguna [2], Kumar et al. [9] and Pazy [12].

**PRELIMNARIES**

Let X denote a Banach space and let J denote the closure of the interval $[t_0, T)$, $t_0 < T < \infty$. We assume that $-A$ be the infinitesimal generator of an analytic semi-group $T(t)$ on the Banach space X . For convenience we also assume that $T(t)$ is bounded, i.e. $\|T(t)\| \leq M$ for $t \geq 0$ and $0 \in \rho(-A)$ i.e. $-A$ is invertible. We note that if $-A$ is the infinitesimal generator of an analytic semi-group then $-(A + \alpha I)$ is invertible and generates a bounded analytic semi-group for $\alpha > 0$ sufficiently large. This enables one to reduce the general case where $-A$ is the infinitesimal generator of an analytic semigroup to the case in which the semi-group is bounded and $-A$ is invertible. It follows that A^α can be defined for $0 \leq \alpha \leq 1$ and A^α is a closed linear invertible operator with domain $D(A^\alpha)$ dense in X . The closedness of A^α implies that $D(A^\alpha)$ endowed with the graph norm of A^α , i.e. the norm $\|x\| = \|x\| + \|A^\alpha x\|$, is a Banach Space. Since A^α is invertible and its graph norm $\| \cdot \|$ is equivalent to the norm $\|x\|_\alpha = \|A^\alpha x\|$, which is equivalent to the graph norm of A^α . We have $X_\beta \rightarrow X_\alpha$ for $0 < \alpha < \beta$ and the embedding is continuous.

By a classical solution to (1.1) – (1.2) on J , we mean a function $u \in C(J; X) \cap (J \setminus \{t_0\}; X)$ satisfying (1.1) – (1.2) on J . By a local classical solution to (1.1) – (1.2) on J , we mean that there exist a $T_0, t_0 < T_0 < T$, and a function u defined from $J_0 = [t_0, T_0]$ into X such that u is a classical solution to (1.1) – (1.2) on J_0 .

Definition 2.1: A continuous function $u: J \rightarrow X$ is said to be a mild solution of problem (1.1) – (1.2) if for all $u_0 \in X$, it satisfies the integral equation

$$u(t) = T(t - t_0)u_0 + \int_{t_0}^t T(t-s) \left[f \left(s, u(s), \int_{t_0}^s k(s, \tau, u(\tau)) d\tau, \int_{t_0}^s h(s, \tau, u(\tau)) d\tau \right) + \int_{t_0}^s B(s-\tau) g(\tau, u(\tau)) d\tau \right] ds$$

By a mild solution of (1.1) – (1.2) on J , we mean that there exist a $T_0, t_0 < T_0 < T$, and a function u defined from $J_0 = [t_0, T_0]$ into X such that u is a mild solution of (1.1) – (1.2). To state and prove the main results, we shall require the following assumption on the maps f, k, h and g .

(A₁) Let D be an open subset of $[0, \infty) \times X_\alpha \times X_\alpha \times X_\alpha$ and for every $(t, x, y, z) \in D$ there exist a neighbourhood $V \subset D$ of (t, x, y, z) and constants $L_1 \geq 0, 0 < \theta < 1$ such that

$$\|f(s_1, u_1, v_1, w_1) - f(s_2, u_2, v_2, w_2)\| \leq L_1 \left[|s_1 - s_2|^\theta + \|u_1 - u_2\|_\alpha + \|v_1 - v_2\|_\alpha + \|w_1 - w_2\|_\alpha \right]$$

for all $(s_i, u_i, v_i, w_i) \in V, i = 1, 2$.

(A₂) Let E be an open subset of $[0, \infty) \times [0, \infty) \times X_\alpha$ and for every $(t, s, x) \in E$ there exist a neighbourhood $U \subset E$ of (t, s, x) and constants $L_2, L_3 \geq 0, 0 < \theta < 1$ such that

$$\|k(t_1, s_1, u_1) - k(t_2, s_2, u_2)\| \leq L_2 \left[|t_1 - t_2|^\theta + |s_1 - s_2|^\theta + \|u_1 - u_2\|_\alpha \right]$$

$$\|h(t_1, s_1, u_1) - h(t_2, s_2, u_2)\| \leq L_3 \left[|t_1 - t_2|^\theta + |s_1 - s_2|^\theta + \|u_1 - u_2\|_\alpha \right]$$

for all $(t_i, s_i, u_i) \in U, i = 1, 2$.



(A₃) Let P be an open subset of $[0, \infty) \times X_\alpha$ and for every $(t, x) \in P$ there exist a neighborhood $W \subset P$ of (t, x) and constants $L_0 \geq 0, 0 < \theta < 1$ such that

$$\|g(t_1, u_1) - g(t_2, u_2)\| \leq L_0 \left[|t_1 - t_2|^\theta + \|u_1 - u_2\|_\alpha \right]$$

for all $(t_i, u_i) \in W, i = 1, 2$.

LOCAL EXISTANCE OF MILD SOLUTIONS

To establish local existence of the considered problem, we assume without loss of generality that the analytic semigroup generated by $-A$ is bounded and that $-A$ is invertible and $0 < T < \infty$. Now we state the following theorem.

Theorem 3.1: Suppose that $-A$ generates the analytic semigroup $T(t)$ such that $\|T(t)\| \leq M$ and $0 \in \rho(-A)$. If the maps f, k, h and g satisfy assumption (A₁), (A₂), (A₃) and the real valued map B is integrable on J , then (1.1) – (1.2) has a unique local mild solution for every $u_0 \in X_\alpha$.

Proof: Let us fix a point (t_0, u_0) in the open subset D of $[0, \infty) \times X_\alpha$ and choose $t'_1 > t_0$ and $\delta > 0$ such that (A₁), (A₂), (A₃) holds for the functions f, k, h and g on the set

$$V = \{(t, x) \in D : t_0 \leq t \leq t'_1, \|x - u_0\|_\alpha \leq \delta\} \quad (3.1)$$

$$\text{Let } N_1 = \sup_{t_0 \leq t \leq t'_1} \left\| f \left(t, u_0, \int_{t_0}^t k(t, s, u_0) ds, \int_{t_0}^t h(t, s, u_0) ds \right) \right\| \quad \text{and} \quad N_2 = \sup_{t_0 \leq t \leq t'_1} \|g(t, u_0)\|$$

Choose $t_1 > t_0$ such that

$$\|T(t - t_0)A^\alpha u_0 - A^\alpha u_0\| \leq \delta/2, \text{ for } t_0 \leq t \leq t_1 \quad (3.2)$$

$$\text{and } 0 < t_1 - t_0 < \min \left\{ t'_1 - t_0, \left[\frac{\delta}{2} C_\alpha^{-1} (1 - \alpha) \{L_1 \delta + L_2 \delta (t_1 - t_0) + L_3 \delta (t_1 - t_0) + N_1 + a_T (L_0 \delta + N_2)\}^{-1} \right]^{\frac{1}{1-\alpha}} \right\}$$

where C_α is a positive constant depending on α satisfying

$$\|A^\alpha T(t)\| \leq C_\alpha t^{-\alpha} \text{ for } t > t_0 \quad (3.3)$$

and a_T is such that

$$a_T = \int_{t_0}^T |B(s)| ds \quad (3.4)$$

Let $Y = C([t_0, t_1]; X)$ be endowed with the supremum norm $\|y\|_Y = \sup_{t_0 \leq t \leq t_1} \|y(t)\|$, then Y is a Banach space. We define a map on Y by $Fy = \tilde{y}$ where \tilde{y} is given by

$$\tilde{y}(t) = T(t - t_0)A^\alpha u_0(t_0) + \int_{t_0}^t A^\alpha T(t - s) \left[f \left(s, A^{-\alpha} y(s), \int_{t_0}^s k(s, \tau, A^{-\alpha} y(\tau)) d\tau, \int_{t_0}^s h(s, \tau, A^{-\alpha} y(\tau)) d\tau \right) \right. \\ \left. + \int_{t_0}^s B(s - \tau) g(\tau, A^{-\alpha} y(\tau)) d\tau \right] ds$$

Now, for every $y \in Y, Fy(t_0) = A^\alpha u_0(t_0)$ and for $t_0 \leq s \leq t \leq t_1$, we have

$$Fy(t) - Fy(s) = [T(t - t_0) - T(s - t_0)]A^\alpha u_0(t_0) + \int_s^t A^\alpha T(t - \tau) \left[f \left(\tau, A^{-\alpha} y(\tau), \int_{t_0}^\tau k(\tau, \xi, A^{-\alpha} y(\xi)) d\xi, \int_{t_0}^\tau h(\tau, \xi, A^{-\alpha} y(\xi)) d\xi \right) \right. \\ \left. + \int_{t_0}^\tau B(\tau - \mu) g(\mu, A^{-\alpha} y(\mu)) d\mu \right] d\tau + \int_{t_0}^s A^\alpha [T(t - \tau) - T(s - \tau)]$$



$$\left[f \left(\tau, A^{-\alpha} y(\tau), \int_{t_0}^{\tau} k(\tau, \xi, A^{-\alpha} y(\xi)) d\xi, \int_{t_0}^{\tau} h(\tau, \xi, A^{-\alpha} y(\xi)) d\xi \right) + \int_{t_0}^{\tau} B(\tau - \mu) g(\mu, A^{-\alpha} y(\mu)) d\mu \right] d\tau \quad (3.5)$$

It follows from the (A_1) , (A_2) , (A_3) on the functions f, k, h and g , (3.3) and (3.4) that $F : Y \rightarrow Y$. Let S be the nonempty closed and bounded subset of Y defined by

$$S = \left[y \in Y : y(t_0) = A^{\alpha} u_0(t_0), \|y(t) - A^{\alpha} u_0\| \leq \delta \right] \quad (3.6)$$

Then for $y \in S$, we have

$$\begin{aligned} \|Fy(t) - A^{\alpha} u_0\| &\leq \|T(t-t_0) - I\| \|A^{\alpha} u_0\| + \int_{t_0}^t \|A^{\alpha} T(t-s)\| \left\| f \left(s, A^{-\alpha} y(s), \int_{t_0}^s k(s, \tau, A^{-\alpha} y(\tau)) d\tau, \int_{t_0}^s h(s, \tau, A^{-\alpha} y(\tau)) d\tau \right) \right. \\ &- f \left(s, u_0, \int_{t_0}^s k(s, \tau, u_0) d\tau, \int_{t_0}^s h(s, \tau, u_0) d\tau \right) \left. \right\| ds + \int_{t_0}^t \|A^{\alpha} T(t-s)\| \left\| \int_{t_0}^s B(s-\tau) \|g(\tau, A^{-\alpha} y(\tau)) - g(\tau, u_0)\| d\tau \right\| ds \\ &+ \int_{t_0}^t \|A^{\alpha} T(t-s)\| \left\| f \left(s, u_0, \int_{t_0}^s k(s, \tau, u_0) d\tau, \int_{t_0}^s h(s, \tau, u_0) d\tau \right) + \int_{t_0}^s B(s-\tau) \|g(\tau, u_0)\| d\tau \right\| ds \\ &\leq \frac{\delta}{2} + C_{\alpha} (1-\alpha)^{-1} \{L_1 \delta + L_2 \delta (t_1 - t_0) + L_3 \delta (t_1 - t_0) + N_1 + a_T (L_0 \delta + N_2)\} (t_1 - t_0)^{(1-\alpha)} \leq \delta \end{aligned} \quad (3.7)$$

It follows that $F : S \rightarrow S$. Now we show that F is a strict contraction on S which will ensure the existence of a unique continuous function satisfying the equation (2.1).

Let y and z be in S . Then

$$\begin{aligned} \|Fy(t) - Fz(t)\| &\leq \|\tilde{y}(t) - \tilde{z}(t)\| \\ &\leq \int_{t_0}^t \|A^{\alpha} T(t-s)\| \left\| f \left(s, A^{-\alpha} y(s), \int_{t_0}^s k(s, \tau, A^{-\alpha} y(\tau)) d\tau, \int_{t_0}^s h(s, \tau, A^{-\alpha} y(\tau)) d\tau \right) - f \left(s, A^{-\alpha} z(s), \int_{t_0}^s k(s, \tau, A^{-\alpha} z(\tau)) d\tau, \int_{t_0}^s h(s, \tau, A^{-\alpha} z(\tau)) d\tau \right) \right\| ds \\ &+ \int_{t_0}^t \|A^{\alpha} T(t-s)\| \left\| \int_{t_0}^s B(s-\tau) \|g(\tau, A^{-\alpha} y(\tau)) - g(\tau, A^{-\alpha} z(\tau))\| d\tau \right\| ds \end{aligned} \quad (3.8)$$

Using (A_1) , (A_2) , (A_3) on the functions f, k, h and g , (3.3) and (3.4), we get

$$\begin{aligned} \|Fy(t) - Fz(t)\| &\leq \left[\{L_1 + L_2 (t_1 - t_0) + L_3 (t_1 - t_0) + a_T L_0\} \int_{t_0}^t \|A^{\alpha} T(t-s)\| ds \right] \|y - z\|_Y \\ &\leq \{L_1 + L_2 (t_1 - t_0) + L_3 (t_1 - t_0) + a_T L_0\} C_{\alpha} (1-\alpha)^{-1} (t_1 - t_0)^{1-\alpha} \|y - z\|_Y \\ &\leq \frac{1}{\delta} \{L_1 \delta + L_2 \delta (t_1 - t_0) + L_3 \delta (t_1 - t_0) + N_1 + a_T (L_0 \delta + N_2)\} C_{\alpha} (1-\alpha)^{-1} (t_1 - t_0)^{1-\alpha} \|y - z\|_Y \\ &\leq \frac{1}{2} \|y - z\|_Y \end{aligned} \quad (3.9)$$

Thus F is a strict contraction map from S into S and therefore by the Banach contraction principle there exists a unique fixed point y of F in S , i.e., there is a unique $y \in S$ such that

$$Fy = y = \tilde{y} \quad (3.10)$$

Let $u = A^{-\alpha} y$. Then for $t \in [t_0, t_1]$, we have

$$u(t) = A^{-\alpha} y(t) = T(t-t_0)u_0 + \int_{t_0}^t T(t-s) \left[f \left(s, u(s), \int_{t_0}^s k(s, \tau, u(\tau)) d\tau, \int_{t_0}^s h(s, \tau, u(\tau)) d\tau \right) + \int_{t_0}^s B(s-t) g(\tau, u(\tau)) d\tau \right] ds \quad (3.11)$$

Thus u is unique local mild solution to (1.1) – (1.2).

REGULARITY OF MILD SOLUTIONS

Obtaining a mild solution for the problem under consideration is relatively easy as we are searching the solution in a larger class. The next task is to show, under additional conditions,



that the solution obtained is actually nice in the sense that it has differentiability properties. Establishing this kind of result is said to establish the regularity of the solution.

In this section we establish the regularity of the mild solutions to (1.1) – (1.2). Again, let J denote the closure of the interval $[t_0, T)$, $t_0 < T \leq \infty$. In addition to the hypothesis mentioned in the earlier sections, on the kernel B we assume the following:

(H) There exist constants $L_4 \leq 0$ and $0 < \beta \leq 1$ such that

$$|B(t) - B(s)| \leq L_4 |t - s|^\beta \text{ for all } t, s \in J.$$

Theorem 4.1. Suppose that $-A$ generates the analytic semigroup $T(t)$ such that $\|T(t)\| \leq M$ for $t \geq 0$ and $0 \in -\rho(A)$. Further, suppose that the maps f, k, h and g satisfy the assumption $(A_1), (A_2), (A_3)$ and the kernel B satisfies (H). Then (1.1) – (1.2) has a unique local classical solution.

Proof. From theorem 3.1, it follows that there exist T_0 , $0 < T_0 < T$ and a function u such that u is a unique mild solution to (1.1) – (1.2) on $J_0 = [t_0, T_0)$ given by

$$u(t) = T(t - t_0)u_0(t_0) + \int_{t_0}^t T(t - s) \left[f \left(s, u(s), \int_{t_0}^s k(s, \tau, u(\tau)) d\tau, \int_{t_0}^s h(s, \tau, u(\tau)) d\tau \right) + \int_{t_0}^s B(s - \tau) g(\tau, u(\tau)) d\tau \right] ds, \quad t_0 \in J \quad (4.1)$$

Let
$$v(t) = A^\alpha u(t)$$

(4.2)

Then

$$v(t) = T(t - t_0)A^\alpha u_0(t_0) + \int_{t_0}^t A^\alpha T(t - s) \left[f \left(s, A^{-\alpha} v(s), \int_{t_0}^s k(s, \tau, A^{-\alpha} v(\tau)) d\tau, \int_{t_0}^s h(s, \tau, A^{-\alpha} v(\tau)) d\tau \right) + \int_{t_0}^s B(s - \tau) g(\tau, A^{-\alpha} v(\tau)) d\tau \right] ds \quad (4.3)$$

For simplification, let us denote

$$\tilde{f}(t) = f \left(t, A^{-\alpha} v(t), \int_{t_0}^s k(t, s, A^{-\alpha} v(s)) ds, \int_{t_0}^s h(t, s, A^{-\alpha} v(s)) ds \right), \quad \tilde{g}(t) = g(t, A^{-\alpha} v(t)). \quad (4.4)$$

Then (4.3) can be rewritten as

$$v(t) = T(t - t_0)A^\alpha u_0(t_0) + \int_{t_0}^t A^\alpha T(t - s) \left[\tilde{f}(s) + \int_{t_0}^s B(s - \tau) \tilde{g}(\tau) d\tau \right] ds \quad (4.5)$$

Since $u(t)$ is continuous on J_0 and the maps f, k, h and g satisfy the assumption $(A_1), (A_2), (A_3)$, it follows that \tilde{f} and \tilde{g} are continuous on J_0 and therefore bounded on J_0 . Let

$$B_1 = \sup_{t \in J_0} \|\tilde{f}(t)\|; \quad B_2 = \sup_{t \in J_0} \|\tilde{g}(t)\| \quad (4.6)$$

We show that \tilde{f} and \tilde{g} are locally Hölder continuous on J_0 . For this, we first show that $v(t)$ is locally Hölder continuous on J_0 . From theorem 6.13 in Pazy [12] p. 74, it follows that for every $0 < \beta < 1 - \alpha$ and every $0 < h < 1$, we have

$$\|(T(h) - I)A^\alpha T(t - s)\| \leq C_\beta h^\beta \|A^{\alpha + \beta} T(t - s)\| \leq Ch^\beta (t - s)^{-(\alpha + \beta)} \quad (4.7)$$



Now, we have

$$\begin{aligned} \|v(t+h)-v(t)\| \leq & \left\| (T(h)-I)T(t-t_0)A^\alpha u_0 \right\| + \int_{t_0}^t \left\| (T(h)-I)A^\alpha T(t-s) \right\| \left\| \tilde{f}(s) + \int_{t_0}^s B(s-\tau)\tilde{g}(\tau)d\tau \right\| ds \\ & + \int_t^{t+h} \left\| A^\alpha T(t+h-s) \right\| \left\| \tilde{f}(s) + \int_{t_0}^s B(s-\tau)\tilde{g}(\tau)d\tau \right\| ds \end{aligned} \quad (4.8)$$

Now

$$\left\| (T(h)-I)T(t-t_0)A^\alpha u_0 \right\| \leq C(t-t_0)^{-(\alpha+\beta)} h^\beta \|u_0\| \leq M_1 h^\beta \quad (4.9)$$

where M_1 depends on t and blows up as t decreases to t_0 . Furthermore

$$\int_{t_0}^t \left\| (T(h)-I)A^\alpha T(t-s) \right\| \left\| \tilde{f}(s) + \int_{t_0}^s B(s-\tau)\tilde{g}(\tau)d\tau \right\| ds \leq [B_1 + a_{t_0} B_2 T_0] h^\beta C_\beta \int_{t_0}^t (t-s)^{-(\alpha+\beta)} ds \leq M_2 h^\beta \quad (4.10)$$

where M_2 is independent of t . Also, we have

$$\int_t^{t+h} \left\| A^\alpha T(t+h-s) \right\| \left\| \tilde{f}(s) + \int_{t_0}^s B(s-\tau)\tilde{g}(\tau)d\tau \right\| ds \leq [B_1 + a_{t_0} B_2 T_0] C_\alpha \int_t^{t+h} (t+h-s)^{-\alpha} ds \leq M_3 h^\beta \quad (4.11)$$

where M_3 is independent of t . From the estimates (4.9 – 4.11), it follows that there exists a constant C_1 such that for every $t'_0 > t_0$, we have

$$\|v(t)-v(s)\| \leq C_1 |t-s|^\beta \quad (4.12)$$

for all $t_0 < t'_0 \leq t, s \leq T_0$. Now, the assumption $(A_1), (A_2), (A_3)$ together with (4.12) implies that there exist constants $C_2, C_3 \geq 0$ and $0 < \gamma, \eta < 1$ such that for all $t_0 < t'_0 \leq t, s \leq T_0$, we have

$$\|\tilde{f}(t)-\tilde{f}(s)\| \leq C_2 |t-s|^\gamma, \quad \|\tilde{g}(t)-\tilde{g}(s)\| \leq C_3 |t-s|^\eta \quad (4.13)$$

Let
$$h(t) = \tilde{f}(t) + \int_{t_0}^t B(t-\tau)\tilde{g}(\tau)d\tau \quad (4.14)$$

Now, we show that $h(t)$ is locally Hölder continuous on J_0 . For $s \leq t$, we have

$$\begin{aligned} \|h(t)-h(s)\| &= \|\tilde{f}(t)-\tilde{f}(s)\| + \int_{t_0}^s \|B(t-\tau)-B(s-\tau)\| \|\tilde{g}(\tau)\| d\tau + \int_s^t \|B(s-\tau)\| \|\tilde{g}(\tau)\| d\tau \\ &\leq C_2 |t-s|^\gamma + B_2 L_4 T_0 |t-s|^\beta + B_2 a_{t_0} (2T_0)^{1-\beta} |t-s|^\beta \leq C_4 |t-s|^\delta \end{aligned} \quad (4.15)$$

for some constants $C_4 \leq 0$ and $0 < \delta < 1$. Consider the following initial value problem

$$\begin{aligned} \frac{dv(t)}{dt} + Av(t) &= h(t), \quad t > t_0 \\ v(t_0) &= u_0. \end{aligned} \quad (4.16)$$

By Corollary 3.3 in Pazy [12] p. 112, (4.16) has a unique solution $v \in C^1((t_0, T_0]; X)$ and the solution is given by

$$v(t) = T(t-t_0)u_0(t_0) + \int_{t_0}^t T(t-s)h(s)ds \quad (4.17)$$

For $t > t_0$, each term on the right hand side belongs to $D(A)$ and hence belongs to $D(A^\alpha)$. Applying A^α on both sides of (4.17) and using the uniqueness of $v(t)$, we have that $A^\alpha v(t) = u(t)$. Thus, it follows that u is the classical solution to (1.1) – (1.2) on J_0 .

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