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COUPLED COINCIDENCE POINT AND COUPLED FIXED POINT THEOREMS FOR WEAK GENERALIZED MEIR-KEELER TYPE CONTRACTIONS IN ORDERED PARTIAL METRIC SPACES

[1]K.P.R.Rao, [2]K.R.K.Rao*

[1]Department of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar -522 510, A.P., India.
[2]Department of Mathematics, Vignana Bharathi Institute of Technology , Aushapur(V) , Ghatkesar(M) , Hyderabad–501 301, A.P., India.

*Corresponding author: krkr08@gmail.com
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ABSTRACT
In this paper we introduce weak generalized Meir-Keeler contractions and prove some coupled fixed point theorems for two mappings $F : X \times X \to X$ and $g : X \to X$ on a partially ordered partial metric spaces. Our results generalize some recent results in the literature, for example the results of Thabet Abdeljawad et.al [1] and Ali Erduran et.al [3] . Also, we give some illustrative examples .

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1 INTRODUCTION AND PRELIMINARIES
Fixed point theory is an important tool in the study of nonlinear analysis as it is considered to be the key connection between pure and applied mathematics with wide applications in economics, physical sciences, such as biology, chemistry, physics, differential equations, and almost all engineering fields. In the last years, the extension of the theory of fixed point to generalized structures as cone metric , dislocated metric , partial metric and quasi-metric spaces has received a lot of attention. One of the most interesting is partial metric space was introduced by Matthews [14] as a part of the study of denotational semantics of data flow networks. Subsequently, Valero [23] and Oltra and Valero [16] gave some generalizations of the results of Matthews. Romaguera [18] proved the Caristi type fixed point theorem on this space.

On the other hand, considering the existence and uniqueness of a fixed point in partially ordered sets initiated a new trend in fixed point theory. The first result in this direction was given by Turinici [22], where he extended Banach contraction principle in partially ordered sets. Ran and Reurings [17] presented some applications of Turinici's theorem to matrix equations. Worth mentioning, Gnana Bhaskar and Lakshmikantham [7] introduced the notion of a coupled fixed point in the class of partially ordered metric spaces and also Lakshmikantham and Ciric [12] introduced the concept of g -mixed monotone property and proved some coupled fixed theorems in metric spaces. Later several authors proved the coupled fixed point theorems in metric and partial metric spaces (see e.g [3, 5, 9, 10, 13, 20]).

In recent years , many authors generalized Meir-keeler fixed point theorems in various spaces which include complete metric and partial metric spaces (see e.g [1, 2, 4, 6,19]). Very recently Ali Erduran et. al. [3] proved the results of coupled fixed points for single map by using generalized Meir-keeler contractions in ordered partial metric spaces and then Thabet Abdeljawad et.al. [1] extended the results of [3] for two mappings . In this paper we introduce weak generalized g_p-Meir-Keeler contractions and prove some coupled fixed point theorems for two mappings $F: X \times X \to X$ and $g: X \to X$ on a partially ordered partial metric spaces and generalize the results of [1] with a suitable example. First we recall some basic definitions and lemmas which play crucial role in the theory of partial metric spaces.

Definition 1.1 [14,15]: A partial metric on a nonempty set X is a function $p: X \times X \to R^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p_2) $p(x,x) \le p(x,y), p(y,y) \le p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

The pair (X, p) is called a partial metric space (PMS).

If p is a partial metric on X, then the function $d_p : X \times X \to \mathrm{P}^+$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y), \tag{1.1}$$

is a metric on X .

The basic example of partial metric space is

Example 1.1[14,15]: Consider $X = [0, \infty)$ with $p(x,y) = \max\{x, y\}$. Then (X, p) is a partial metric space. It is clear that p is not a (usual) metric. Note that in this case $d_p(x,y) = |x - y|$.

Example 1.2 [21]: Let (X, d) and (X, p) denote a metric and partial metric space , respectively Then the mapping $\rho: X \times X \to R^+$ defined by

$$\rho(x,y) = d(x,y) + p(x,y)$$

is a partial metric and (X, ρ) is a partial metric space.

Lemma 1.1 [2]: Let (X, p) be a complete PMS. Then

1. If $p(x,y) = 0$ then $x = y$,
2. If $x \neq y$, then $p(x,y) > 0$,
3. If $x = y$, $p(x,y)$ may not be 0.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon), x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

We now state some basic topological notions (such as convergence, completeness, continuity) on partial metric spaces (see e.g. [2, 11, 14,15]).

Definition 1.2 [14,15]:

1. A sequence $\{x_n\}$ in the PMS (X, p) converges to the limit x if and only if
   $p(x,x) = \lim_{n \to \infty} p(x, x_n)$.

2. A sequence $\{x_n\}$ in the PMS (X, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.
3. A PMS (X, p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n, x_m)$.
4. A mapping $F : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $F(B_p(x_0, \delta)) \subseteq B_p(Fx_0, \varepsilon)$.

The following lemma is one of the basic results in PMS([2,11,14,15]).

Lemma 1.2 [14,15]:

1. A sequence $\{x_n\}$ is a Cauchy sequence in the PMS (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p).
2. A PMS (X, p) is complete if and only if the metric space (X, d_p) is complete.

Moreover $\lim_{n\to\infty} d_p(x, x_n) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m)$ (1.2)

Next, we give a simple lemma which will be used in the proofs of our main results. For the proofs we refer to e.g. [2,11].

Lemma 1.3 [2,11]: Assume $x_n \to z$ as $n \to \infty$ in a PMS (X, p) such that $p(z,z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.

The concept of g-mixed monotone property was introduced by Lakshmikantam and Ciric [12] as follows

Definition 1.3 [12]: Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. Then the map F is said to have mixed g - monotone property if $F(x, y)$ is monotone g - non - decreasing in x and is monotone g - non - increasing in y; that is, for any $x, y \in X$,

$$gx_1 \le gx_2 \text{ implies } F(x_1, y) \le F(x_2, y) \text{ for all } y \in X \text{ and}$$
$$gy_1 \le gy_2 \text{ implies } F(x, y_2) \le F(x, y_1) \text{ for all } x \in X.$$

Definition 1.4 [12]: An element $(x, y) \in X \times X$ is called

(g_1) a *coupled coincident point* of mappings $F : X \times X \to X$ and $f : X \to X$ if $fx = F(x, y)$ and $fy = F(y, x)$.

(g_2) a *common coupled fixed point* of mappings $F : X \times X \to X$ and $f : X \to X$ if $x = fx = F(x, y)$ and $y = fy = F(y, x)$.

The mappings F and g are said to commute if $g(F(x, y)) = F(g(x), g(y))$ *for all* $x, y, \in X$

Recently, Gordji et.al. [8] replaced mixed g-monotone property with a mixed strict g-monotone property and improved the results of [12] .

Definition 1.5 [8]: Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. Then the map F is said to have mixed strict g - monotone property if $F(x, y)$ is monotone g - non - decreasing in x and is monotone g - non - increasing in y; that is, for any $x, y \in X$,

$$gx_1 < gx_2 \text{ implies } F(x_1, y) < F(x_2, y) \text{ for all } y \in X$$

and

$$gy_1 < gy_2 \text{ implies } F(x, y_1) > F(x, y_2) \text{ for all } x \in X.$$

Very recently, Thabet Abdeljawad et.al. [1] Introduced the following g-meir-keeler type

and strict g -meir-keeler type contraction as follows :

Definition 1.6 [1]: Let $(X, p, \leq)$ be a partial metric space. Let $F : X \times X \to X$ and $g : X \to X$. The mapping F is said to be a g - meir-keeler type contraction if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \frac{1}{2}\left[p(g(x), g(u)) + p(g(y), g(v))\right] < \varepsilon + \delta(\varepsilon) \Rightarrow p(F(x, y), F(u, v)) < \varepsilon, \tag{1.3}$$

for all $x, y, u, v \in X$ with $g(x) \leq g(u)$, $g(y) \geq g(v)$.

If $g = I_X$, identity map in the Definition 1.6, gives the following definition of [3].

Definition 1.7 [3]: Let $(X, p, \leq)$ be a partial metric space. Let $F : X \times X \to X$ be a given mapping. we say that F is said to be a generalized Meir-Keeler type contraction if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \frac{1}{2}\left[p(x, u) + p(y, v)\right] < \varepsilon + \delta(\varepsilon) \Rightarrow p(F(x, y), F(u, v)) < \varepsilon, \tag{1.4}$$

for all $x, y, u, v \in X$ with $x \leq u$, $y \geq v$.

Definition 1.8 [1]: Let $(X, p, \leq)$ be a partial metric space. Let $F : X \times X \to X$ and $g : X \to X$. The mapping F is said to be a strict g - Meir-Keeler type contraction if there exists $0 < k < 1$ such that for any $\varepsilon > 0$ $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \frac{k}{2}\left[p(g(x), g(u)) + p(g(y), g(v))\right] < \varepsilon + \delta(\varepsilon) \Rightarrow p(F(x, y), F(u, v)) < \varepsilon, \tag{1.5}$$

for all $x, y, u, v \in X$ with $g(x) \leq g(u)$, $g(y) \geq g(v)$.

Based on the Definition 1.6 , Thabet Abdeljawad et.al. [1] proved the folowing results.

Theorem 1.1 [1]: Let $(X, p, \leq)$ be a partially ordered partial metric space. Suppose that X has the following properties:

(a) If $\{x_n\}$ is a sequence such that $x_{n+1} > x_n$ for each $n = 1,2,3\cdots$ and $x_n \to x$, then $x_n < x$ for each $n = 1,2,3\cdots$.

(b) If $\{y_n\}$ is a sequence such that $y_{n+1} < y_n$ for each $n = 1,2,3\cdots$ and $y_n \to y$, then $y_n > y$ for each $n = 1,2,3\cdots$.

Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of (X, p). Suppose that F satisfies the following conditions:

(i) F has mixed strict g - monotone property,

(ii) F is a g -Meir-Keeler type contraction ,

(iii) There exist x_0, $y_0 \in X$ such that $gx_0 < F(x_0, y_0)$, $gy_0 \geq F(y_0, x_0)$.

Then F and g have a coupled coincidence point, that is there exist $x, y \in X$ such that $F(x, y) = g(x)$, $F(y, x) = g(y)$.

Theorem 1.2 [1]: In addition to the hypotheses of Theorem 1.1 , assume that for all $(x, y), (x^*, y^*) \in X^2$, there exists $(a, b) \in X^2$ such that $(F(a, b), F(b, a))$ is comparable to both $(g(x), g(y))$ and $(g(x^*), g(y^*))$. Further , assume that F and g commute and F is a strict g -Meir-Keeler type contraction. Then F and g have a unique common coupled fixed point, that is: $x = g(x) = F(x, y), \quad y = g(y) = F(y, x).$ (1.6)

Now, we give the following definitions.

Definition 1.9: Let $(X, p, \leq)$ be a partial metric space. Let $F : X \times X \to X$ and $g : X \to X$. The mapping F is said to be a weak generalized g_p - Meir-Keeler type contraction if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \max\{p(g(x), g(u)), p(g(y), g(v))\} < \varepsilon + \delta(\varepsilon) \Rightarrow \max\begin{Bmatrix} p(F(x,y), F(u,v)), \\ p(F(y,x), F(v,u)) \end{Bmatrix} < \varepsilon, \tag{1.7}$$

for all $x, y, u, v \in X$ with $g(x) < g(u)$, $g(y) > g(v)$.

If we replace $g = I_X$, identity map then, we have

Definition 1.10: Let $(X, p, \leq)$ be a partial metric space. Let $F : X \times X \to X$. The mapping F is said to be a weak generalized Meir-Keeler type contraction if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \max\{p(x,u), p(y,v)\} < \varepsilon + \delta(\varepsilon) \Rightarrow \max\begin{Bmatrix} p(F(x,y), F(u,v)), \\ p(F(y,x), F(v,u)) \end{Bmatrix} < \varepsilon, \tag{1.8}$$

for all $x, y, u, v \in X$ with $x < u$, $y > v$.

2 MAIN RESULTS

First we give the following Lemma which can be derived easily from Definition 1.9 and which is essential in proving our main result.

Lemma 2.1: Let $(X, p, \leq)$ be a partially ordered partial metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F is a weak generalized g_p - Meir-Keeler type contraction. Then $\max\{p(F(x,y), F(u,v)), p(F(y,x), F(v,u))\} < \max\{p(g(x), g(u)), p(g(y), g(v))\}$ for all $x, y, u, v \in X$ with $g(x) < g(u)$, $g(y) > g(v)$.

Now, we give our main result.

Theorem 2.1: Let $(X, p, \leq)$ be a partially ordered partial metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of (X, p). Suppose that F satisfies the following conditions:

(2.1.1) F has mixed strict g - monotone property,

(2.1.2) F is a weak generalized g_p -Meir-Keeler type contraction ,

(2.1.3) There exist x_0 , $y_0 \in X$ such that $gx_0 < F(x_0, y_0)$, $gy_0 > F(y_0, x_0)$.

Suppose that X has the following properties:

(2.1.4) If $\{x_n\}$ is a sequence such that $x_{n+1} > x_n$ for each $n = 1,2,3\cdots$ and $x_n \to x$, then $x_n < x$ for each $n = 1,2,3\cdots$.

(2.1.5) If $\{y_n\}$ is a sequence such that $y_{n+1} < y_n$ for each $n = 1,2,3\cdots$ and $y_n \to y$, then $y_n > y$ for each $n = 1,2,3\cdots$.

Then F and g have a coupled coincidence point, that is there exist $x, y \in X$ such that $F(x, y) = g(x)$, $F(y, x) = g(y)$.

Proof: Let $(x, y) = (x_0, y_0) \in X^2$ be such that $g(x_0) < F(x_0, y_0)$ and $g(y_0) > F(y_0, x_0)$. We construct the sequences $\{x_n\}$ and $\{y_n\}$ in the following way.

Since $F(X \times X) \subseteq g(X)$, we are able to choose $(x_1, y_1) \in X^2$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. By repeating the same argument, we can choose $(x_2, y_2) \in X^2$ such that $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$.

Inductively, we construct the sequnces $\{x_n\}$ and $\{y_n\}$ such that

$$g(x_{n+1}) = F(x_n, y_n) \ , \quad g(y_{n+1}) = F(y_n, x_n) \qquad \forall \ n = 1,2,\cdots \tag{2.1}$$

By assumption (2.1.3), we have

$$g(x_0) < F(x_0, y_0) = g(x_1) \ and \ g(y_0) > F(y_0, x_0) = g(y_1) \tag{2.2}$$

Scince F is a mixed g-strict monotone property , we have

$$g(x_0) < g(x_1) \Rightarrow F(x_0, y_0) < F(x_1, y_0). \tag{2.3}$$

$$g(y_0) > g(y_1) \Rightarrow F(x_1, y_0) < F(x_1, y_1). \tag{2.4}$$

Thus

$$g(x_1) = F(x_0, y_0) < F(x_1, y_1) = g(x_2)$$

Also

$$g(x_1) > g(x_0) \Rightarrow F(y_0, x_0) > F(y_0, x_1). \tag{2.5}$$

$$g(y_1) < g(y_0) \Rightarrow F(y_1, x_1) < F(y_0, x_1). \tag{2.6}$$

Thus

$$g(y_1) = F(y_0, x_0) > F(y_1, x_1) = g(y_2).$$

Continuing in this way, we get

$$g(x_0) < g(x_1) < g(x_2) < \cdots < g(x_n) < g(x_{n+1}) < \cdots$$

$$g(y_0) > g(y_1) > g(y_2) > \cdots > g(y_n) > g(y_{n+1}) > \cdots$$

Now set

$$R_n = \max\{p(g(x_n), g(x_{n+1})), p(g(y_n), g(y_{n+1}))\}. \tag{2.7}$$

From Lemma 2.1 , we have

$$\max\begin{Bmatrix} p(g(x_n), g(x_{n+1})), \\ p(g(y_n), g(y_{n+1})) \end{Bmatrix} = \max\begin{Bmatrix} p(F(x_{n-1}, y_{n-1}), F(x_n, y_n)), \\ p(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \end{Bmatrix}$$
$$< \max\{p(g(x_n), g(x_{n+1})), p(g(y_n), g(y_{n+1}))\}.$$

Thus we obtain $R_n < R_{n-1}$.

Hence $\{R_n\}$ is a monotone decreasing sequence in R^+. Since the sequence $\{R_n\}$ is bounded below, there exists $r \geq 0$ such that $\lim_{n\to\infty} R_n = r.$

Suppose that $r > 0$. Then for positive integer k , we have

$$r \leq R_k = \max\{p(g(x_k), g(x_{k+1})), p(g(y_k), g(y_{k+1})) < r + \delta(r). \tag{2.8}$$

Since F is generalized g_p-Meir-Keeler type contraction , we have

$$\max\{p(F(x_k,y_k),F(x_{k+1},y_{k+1})),\ p(F(y_k,x_k),F(y_{k+1},x_{k+1}))\} < r, \tag{2.9}$$

which is equivalent to

$$\max\{p(g(x_{k+1}),g(x_{k+2})),\ p(g(y_{k+1}),g(y_{k+2}))\} < r. \tag{2.10}$$

Hence we obtain, $R_{k+1} < r.$

It is a contradiction. Hence we have $r = 0$. Thus

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \max\{p(g(x_n),g(x_{n+1})),\ p(g(y_n),g(y_{n+1}))\} = 0. \tag{2.11}$$

Consequently, we have

$$\lim_{n\to\infty} p(g(x_n),g(x_{n+1})) = 0 = \lim_{n\to\infty} p(g(y_n),g(y_{n+1})) \tag{2.12}$$

By condition (p_2) and (2.12), we have

$$\lim_{n\to\infty} p(g(x_n),g(x_n)) = 0 \tag{2.13}$$

and

$$\lim_{n\to\infty} p(g(y_n),g(y_n)) = 0. \tag{2.14}$$

We claim that the sequences $\{g(x_n)\}$ and $\{g(x_n)\}$ are Cauchy in $(g(X),p)$.

Take an arbitrary $\varepsilon > 0$. It follows from (2.12) that there exists $k \in N$ such that ,

$$\max\{p(g(x_k),g(x_{k+1})),\ p(g(y_k),g(y_{k+1}))\} < \delta(\varepsilon). \tag{2.15}$$

With out loss of generality , assume that , $\delta(\varepsilon) \le \varepsilon$ and define the following set

$$A = \{(x,y) \in X^2 / \max\{p(x,g(x_k)),\ p(y,g(y_k))\} < \varepsilon + \delta(\varepsilon) \text{ and } x > g(x_k),\ y < g(y_k)\} \tag{2.16}$$

Take $B = (g(X),g(X)) \cap A$. We claim that

$$(F(\alpha,\beta),F(\beta,\alpha)) \in B \quad \forall \quad (x,y) = (g(\alpha),g(\beta)) \in B \tag{2.17}$$

where $\alpha,\beta \in X$. Let $(x,y) = (g(\alpha),g(\beta)) \in B$ then by (2.15) and the triangular inequality , we have

$$\begin{aligned}
\max\begin{Bmatrix} p(g(x_k),F(\alpha,\beta)), \\ p(g(y_k),F(\beta,\alpha)) \end{Bmatrix} &\le \max\begin{Bmatrix} p(g(x_k),g(x_{k+1})) + p(g(x_{k+1}),F(\alpha,\beta)), \\ p(g(y_k),g(y_{k+1})) + p(g(y_{k+1}),F(\beta,\alpha)) \end{Bmatrix} \\
&\le \max\begin{Bmatrix} p(g(x_k),g(x_{k+1})), \\ p(g(y_k),g(y_{k+1})) \end{Bmatrix} + \max\begin{Bmatrix} p(g(x_{k+1}),F(\alpha,\beta)), \\ p(g(y_{k+1}),F(\beta,\alpha)) \end{Bmatrix} \\
&< \delta(\varepsilon) + \max\{p(F(x_k,y_k),F(\alpha,\beta)),\ p(F(y_k,x_k),F(\beta,\alpha))\}.
\end{aligned} \tag{2.18}$$

We distinguish two cases ,

Case (a) : $\max\{p(x,g(x_k)),\ p(y,g(y_k))\} = \max\{p(g(\alpha),g(x_k)),\ p(g(\beta),g(y_k))\} \le \varepsilon$.

By Lemma 2.1 , and the definition of A, the inequalities (2.18) turns into

$$\max\begin{Bmatrix} p(g(x_k),F(\alpha,\beta)), \\ p(g(y_k),F(\beta,\alpha)) \end{Bmatrix} < \delta(\varepsilon) + \max\{p(g(x_k),g(\alpha)),\ p(g(y_k),g(\beta))\}$$

$$\le \delta(\varepsilon) + \varepsilon. \tag{2.19}$$

Case (b): $\varepsilon < \max\begin{Bmatrix} p(x,g(x_k)), \\ p(y,g(y_k)), \end{Bmatrix} = \max\begin{Bmatrix} p(g(\alpha),g(x_k)), \\ p(g(\beta),g(y_k)) \end{Bmatrix} < \varepsilon + \delta(\varepsilon)$

In this case, we have

$$\varepsilon < \max\{p(g(\alpha),g(x_k)),\ p(g(\beta),g(y_k))\} < \varepsilon + \delta(\varepsilon) \tag{2.20}$$

Since $x = g(\alpha) > g(x_k)$ and $y = g(\beta) \le g(y_k)$ by (2.1.2) , we get

$$\max\{p(F(x_k, y_k), F(\alpha, \beta)), p(F(y_k, x_k), F(\beta, \alpha))\} < \varepsilon \tag{2.21}$$

Now by using (2.18) and (2.21) , we have

$$\max\{p(g(x_k), F(\alpha, \beta)), p(g(y_k), F(\beta, \alpha))\} < \varepsilon + \delta(\varepsilon). \tag{2.22}$$

On the other hand, using (2.1.1) , it is obvious that

$$F(\alpha, \beta) > F(x_k, y_k) = g(x_{k+1}) > g(x_k) \ , \quad F(\beta, \alpha) < F(y_k, x_k) = g(y_{k+1}) < g(y_k).$$

So we have $(F(\alpha, \beta), F(\beta, \alpha)) \in A$.

Since $F(X \times X) \subset g(X)$, so $(F(\alpha, \beta), F(\beta, \alpha)) \in B$.

That is (2.17) holds.

On the other hand by (2.15) , we have $(g(x_{k+1}), g(y_{k+1})) \in B$.

This implies with (2.17) that,

$$\begin{aligned} (g(x_{k+1}), g(y_{k+1})) \in B &\Rightarrow (F(x_{k+1}, y_{k+1}), F(y_{k+1}, x_{k+1})) = (g(x_{k+2}), g(y_{k+2})) \in B \\ &\Rightarrow (F(x_{k+2}, y_{k+2}), F(y_{k+2}, x_{k+2})) = (g(x_{k+3}), g(y_{k+3})) \in B \\ &\cdots \Rightarrow (g(x_n), g(y_n)) \in B \Rightarrow \cdots \end{aligned} \tag{2.23}$$

Then, for all $n > k$, we have $(g(x_n), g(y_n)) \in B$. This implies that for all $n, m > k$, we have

$$\begin{aligned} \max\begin{Bmatrix} p(g(x_n), g(x_m)), \\ p(g(y_n), g(y_m)) \end{Bmatrix} &\le \max\begin{Bmatrix} p(g(x_n), g(x_k)) + p(g(x_k), g(x_m)), \\ p(g(y_n), g(y_k)) + p(g(y_k), g(y_m)) \end{Bmatrix} \\ &\le \max\begin{Bmatrix} p(g(x_n), g(x_k)), \\ p(g(y_n), g(y_k)) \end{Bmatrix} + \max\begin{Bmatrix} p(g(x_k), g(x_m)), \\ p(g(y_k), g(y_m)) \end{Bmatrix} \\ &\le 2(\varepsilon + \delta(\varepsilon)) \le 4\varepsilon. \end{aligned}$$

Thus, we have $p(g(x_n), g(x_m)) \le 4\varepsilon$, $p(g(y_n), g(y_m)) \le 4\varepsilon$.

Thus the sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy in $(g(X), p)$. Hence by Lemma 1.2, $\{g(x_n)\}$ and $\{g(y_n)\}$ are also Cauchy sequences in $(g(X), d_p)$.

So

$$\lim_{n\to\infty} d_p(g(x_n), g(x_m)) = 0 \ and \ \lim_{n\to\infty} d_p(g(y_n), g(y_m)) = 0. \tag{2.24}$$

By using the definition of d_p , (2.13) and (2.14), we get

$$\lim_{n\to\infty} p(g(x_n), g(x_m)) = 0 \ and \ \lim_{n\to\infty} p(g(y_n), g(y_m)) = 0. \tag{2.25}$$

Since $(g(X), p)$ is complete , again by Lemma 1.2, we have $(g(X), d_p)$ is complete. So there exist $x, y \in X$ such that $g(x_n) \to gx$ and $g(y_n) \to gy$.

Thus, we have

$$\lim_{n\to\infty} d_p(g(x), g(x_n)) = 0 = \lim_{n\to\infty} d_p(g(y), g(y_n)) \tag{2.26}$$

By using Lemma 1.2 and (2.25), we get

$$p(g(x), g(x)) = \lim_{n\to\infty} p(g(x), g(x_n)) = \lim_{n\to\infty, m\to\infty} p(g(x_n), g(x_m)) = 0. \tag{2.27}$$

$$p(g(y), g(y)) = \lim_{n\to\infty} p(g(y), g(y_n)) = \lim_{n\to\infty, m\to\infty} p(g(y_n), g(y_m)) = 0. \tag{2.28}$$

Since the sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ are monotone increasing and monotone decreasing , respectively , by properties (2.1.4) and (2.1.5) , we conclude that

$$g(x_n) < g(x) \quad , \quad g(y_n) > g(y) \qquad \textit{for each} \qquad n \geq 0.$$

Now by using Lemma 2.1 , we get

$$\max\begin{Bmatrix} p(g(x_{n+1}), F(x,y)), \\ p(g(y_{n+1}), F(y,x)) \end{Bmatrix} = \max\begin{Bmatrix} p(F(x_n, y_n), F(x,y)), \\ p(F(y_n, x_n), F(y,x)) \end{Bmatrix}$$
$$< \max\{p(g(x_n), g(x)), p(g(y_n), g(y))\}.$$

Letting $n \to \infty$ and by using Lemma 1.3, (2.27) and (2.28), we have

$\max\{p(g(x), F(x,y)), p(g(y), F(y,x))\} \leq 0.$ Thus $g(x) = F(x,y)$ and $g(y) = F(y,x).$

Thus $(x,y) \in X^2$ is a coupled coincidence point of F and g .

Corollary 2.1: Let $(X, p, \leq)$ be a complete partially ordered partial metric space. Let $F : X \times X \to X$ be a given mapping. Suppose that F satisfies the following conditions:

(2.1.1.1) F has mixed strict monotone property,

(2.1.1.2) F is a weak generalized Meir-Keeler type contraction ,

(2.1.1.3) There exist x_0 , $y_0 \in X$ such that $x_0 < F(x_0, y_0)$, $y_0 > F(y_0, x_0)$.
Suppose that X has the following properties:

(2.1.1.4) If $\{x_n\}$ is a sequence such that $x_{n+1} > x_n$ for each $n = 1,2,3\cdots$ and $x_n \to x$, then $x_n < x$ for each $n = 1,2,3\cdots$.

(2.1.1.5) If $\{y_n\}$ is a sequence such that $y_{n+1} < y_n$ for each $n = 1,2,3\cdots$ and $y_n \to y$, then $y_n < y$ for each $n = 1,2,3\cdots$.

Then F has a coupled coincidence point, that is there exist $x, y \in X$ such that $F(x,y) = x$, $F(y,x) = y$.

Proof: It follows by taking $g = I_X$, the identity mapping on X , in Theorem 2.1.

Remark 2.1: *In view of weak generalized g_p -Meir-Keeler contraction , Theorem 2.1 is a generalization of Theorem 1.1 and Corollary 2.1 is a generalization of Theorem 1.5 of Ali Erduran et. al.* [3].

The following example illustrates that Theorem 2.1 is more general than Theorem 1.1 .

Example 2.1: Let $X = [0, \infty)$ be endowed with the partial metric $p : X \times X \to [0, \infty)$ defined by $p(x,y) = |x - y| + \max\{x, y\}$ *for all* $x, y \in X$.

Then it is easy to check (X, p) is a complete partial metric space .

Let $g : X \to X$ and $F : X \times X \to X$ be defined as

$$g(x) = x^3 \qquad \textit{and} \qquad F(x,y) = \frac{x^3 + 5y^3}{8}.$$

Then, the mapping F has the strict mixed monotone property. And for $x = 0$, $y = 1$, the condition (2.1.3) of Theorem 2.1 is satisfied. We claim that condition (2.1.2) holds, but the condition (1.3) is not satisfied.

Suppose, to the contrary, that the condition (1.3) is holds.

Then for given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \le \frac{1}{2}\left[p(g(x),g(u)) + p(g(y),g(v))\right] < \delta(\varepsilon) \Rightarrow p(F(x,y),F(u,v)) < \varepsilon, \tag{2.29}$$

for all $x, y, u, v \in X$ with $gx \le gu$, $gy \ge gv$.

$$\begin{aligned} i.e \qquad \varepsilon &\le \frac{1}{2}\left[\left|x^3 - u^3\right| + \max\left\{x^3, u^3\right\} + \left|y^3 - v^3\right| + \max\left\{y^3, v^3\right\}\right] < \varepsilon + \delta(\varepsilon) \\ &\Rightarrow \left|\frac{x^3 + 5y^3}{8} - \frac{u^3 + 5v^3}{8}\right| + \max\left\{\frac{x^3 + 5y^3}{8}, \frac{u^3 + 5v^3}{8}\right\} < \varepsilon. \end{aligned} \tag{2.30}$$

Let $x = u = 0$ and $gy \ge gv$, we get

$$\varepsilon \le \frac{1}{2}\left[\left|y^3 - v^3\right| + y^3\right] < \varepsilon + \delta(\varepsilon). \tag{2.31}$$

Now

$$p(F(x,y),F(u,v)) = \frac{5}{8}\left[\left|(y^3 - v^3)\right| + y^3\right] > \frac{5}{8} \times 2\varepsilon > \varepsilon.$$

It is a contradiction to (2.29). Hence condition (1.3) does not hold.
But F satisfies the condition (2.1.2) . Let $gx < gu$ and $gy > gv$, and

$$\varepsilon \le \max\left\{p(gx,gu), p(gy,gv)\right\} < \varepsilon + \delta(\varepsilon).$$

Then $$\varepsilon \le \max\left\{\left|x^3 - u^3\right| + \max\left\{x^3, u^3\right\}, \left|y^3 - v^3\right| + \max\left\{y^3, v^3\right\}\right\} < \varepsilon + \delta(\varepsilon)$$

which gives

$$\varepsilon \le \max\left\{\left|x^3 - u^3\right| + u^3, \left|y^3 - v^3\right| + y^3\right\} < \varepsilon + \delta(\varepsilon). \tag{2.32}$$

And also, we have

$$p(F(x,y),F(u,v)) = \left|\frac{x^3 + 5y^3}{8} - \frac{u^3 + 5v^3}{8}\right| + \max\left\{\frac{x^3 + 5y^3}{8}, \frac{u^3 + 5v^3}{8}\right\}$$

$$\le \left[\frac{1}{8}\left|x^3 - u^3\right| + \frac{5}{8}\left|y^3 - v^3\right|\right] + \left[\frac{u^3 + 5y^3}{8}\right]$$

$$= \frac{1}{8}\left[\left|x^3 - u^3\right| + u^3\right] + \frac{5}{8}\left[\left|y^3 - v^3\right| + y^3\right]$$

Similarly, we get

$$p(F(y,x),F(v,u)) \le \frac{5}{8}\left[\left|x^3 - u^3\right| + u^3\right] + \frac{1}{8}\left[\left|y^3 - v^3\right| + y^3\right]$$

Hence, we have

$$\max\left\{\begin{matrix} p(F(x,y),F(u,v)), \\ p(F(y,x),F(v,u)) \end{matrix}\right\} \le \max\left\{\begin{matrix} \frac{1}{8}\left[\left|x^3 - u^3\right| + u^3\right] + \frac{5}{8}\left[\left|y^3 - v^3\right| + y^3\right], \\ \frac{5}{8}\left[\left|x^3 - u^3\right| + u^3\right] + \frac{1}{8}\left[\left|y^3 - v^3\right| + y^3\right] \end{matrix}\right\}$$

Without loss of generality, assume that $\left[\left|y^3 - v^3\right| + y^3\right] \le \left[\left|x^3 - u^3\right| + u^3\right]$.

Then $\max\left\{p(F(x,y),F(u,v)), p(F(y,x),F(v,u))\right\} \le \frac{6}{8}\left[\left|x^3 - u^3\right| + u^3\right] < \frac{3}{4}(\varepsilon + \delta(\varepsilon))$.

Thus, by choosing $\delta(\varepsilon) < \frac{\varepsilon}{3}$, the condition (2.1.2) is satisfied.

Thus all the conditions of Theorem 2.1 are satisfied and (0,0) is the coupled coincidence point of F and g.

3 UNIQUENESS OF COUPLED FIXED POINTS

In this section we will prove the uniqueness of a common coupled fixed point. We endow the product space X^2 with the following partial order:

$$(u,v) \le (x,y) \Leftrightarrow u \le x, y \le v, \quad \forall\ (x,y),(u,v) \in X \times X. \tag{3.1}$$

Note that a pair $(x,y) \in X^2$ is comparable with $(u,v) \in X^2$ if either $(x,y) \le (u,v)$ or $(u,v) \le (x,v)$. We next state the conditions for the existence and uniqueness of a common coupled fixed point of maps F and g.

Theorem 3.1: In addition to the hypotheses of Theorem 2.1 , assume that for all $(x,y),(x^*,y^*) \in X^2$, there exists $(a,b) \in X^2$ such that $(F(a,b),F(b,a))$ is comparable to both $(F(x,y),F(y,x))$ and $(F(x^*,y^*),F(y^*,x^*))$. Further , assume that F and g commute ,Then F and g have a unique common coupled fixed point, that is:

$$x = g(x) = F(x,y), \qquad y = g(y) = F(y,x). \tag{3.2}$$

Proof: The set of coupled coincidence points of F and g is not empty due to Theorem 2.1. We suppose that $(x,y),(x^*,y^*) \in X^2$ are two coupled coincidence points of F and g. We distinguish the following two cases.

First Case. $(F(x,y),F(y,x))$ is comparable to $(F(x^*,y^*),F(y^*,x^*))$ with respect to the ordering in X^2, where

$$F(x,y) = g(x), \quad F(y,x) = g(y), \quad F(x^*,y^*) = g(x^*), \quad F(y^*,x^*) = g(y^*). \tag{3.3}$$

Without loss of the generality, we may assume that

$$g(x) = F(x,y) < F(x^*,y^*) = g(x^*), \quad g(y) = F(y,x) > F(y^*,x^*) = g(y^*) \tag{3.4}$$

Now by using Lemma 2.1 , we get

$$\begin{aligned} \max\{p(g(x),g(x^*)), p(g(y),g(y^*))\} &= \max\{p(F(x,y)),F(x^*,y^*)), p(F(y,x)),F(y^*,x^{*^*}))\} \\ &< \max\{p(g(x),g(x^*)), p(g(y),g(y^*)\} \end{aligned} \tag{3.5}$$

which is a contradiction. Therefore, we have $g(x) = g(x^*)$ and $g(y) = g(y^*)$.

Second Case. Suppose that $(F(x,y),F(y,x))$ and $(F(x^*,y^*),F(y^*,x^*))$ are not comparable.

By assumption there exists $(a,b) \in X^2$ such that $(F(a,b),F(b,a))$ is comparable to both $(F(x,y),F(y,x))$ and $(F(x^*,y^*),F(y^*,x^*))$.

Setting $a = a_0$, $b = b_0$, as in the proof of Theorem 2.1, we define the sequences $\{g(a_n)\}$ and $\{g(b_n)\}$ as follows:

$$g(a_{n+1}) = F(a_n,b_n) \quad and \quad g(b_{n+1}) = F(b_n,a_n) \quad \forall \quad n = 0,1,2,\cdots. \tag{3.6}$$

Since $(F(x,y),F(y,x)) = (g(x),g(y))$ and $(F(a,b),F(b,a)) = (g(a_1),g(b_1))$ are comparable,

we may assume without loss of generality that $g(x) < g(a_1)$ and $g(y) > g(b_1)$.
Inductively,we observe that $g(x) < g(a_n)$ and $g(y) > g(b_n)$ $\forall$ $n = 0,1,2,\cdots$.
Thus, by Lemma 2.1, we get that

$$\max\{p(g(x),g(a_{n+1})),p(g(y),g(b_{n+1}))\} = \max\begin{Bmatrix} p(F(x,y)),F(a_n,b_n)),\\ p(F(y,x),F(b_n,b_n)) \end{Bmatrix} \tag{3.7}$$
$$< \max\{p(g(x),g(a_n)),p(g(y),g(b_n))\}$$

Set $\Delta_n = \max\{p(g(x),g(a_n)),p(g(y),g(b_n))\}$. Hence, for each $n \geq 0$

$$\Delta_{n+1} < \Delta_n. \tag{3.8}$$

Therefore, the sequence $\{\Delta_n\}$ is decreasing and bounded below . Hence, it converges to some $s \geq 0$. Assume that $s > 0$. Then, for some positive integer k, we have

$$\varepsilon \leq \Delta_k = \max\{p(g(x),g(a_k)),p(g(y),g(b_k))\} < \varepsilon + \delta(\varepsilon). \tag{3.9}$$

Since F is a weak generalized g_p -Meir-Keeler contraction, we have

$$\max\{p(F(x,y),F(a_k,b_k)),p(F(y,x),F(b_k,a_k))\} < \varepsilon. \tag{3.10}$$

which is equivalent to

$$\max\{p(g(x),g(a_{k+1})),p(g(y),g(b_{k+1}))\} < \varepsilon. \tag{3.11}$$

Hence, we get $\Delta_{k+1} < \varepsilon$.
which is a contradiction . Thus, we deduce that $s = 0$, that is:

$$\lim_{n\to\infty} \max\{p(g(x),g(a_n)),p(g(y),g(b_n))\} = 0. \tag{3.12}$$

In a similar manner, we can show that

$$\lim_{n\to\infty} \max\{p(g(x^*),g(a_n)),p(g(y^*),g(b_n))\} = 0. \tag{3.13}$$

By the triangle inequality, we have

$$\max\begin{Bmatrix} p(g(x^*),g(x)),\\ p(g(y^*),g(y)) \end{Bmatrix} \leq \max\begin{Bmatrix} p(g(x),g(a_n)),\\ p(g(y),g(b_n)) \end{Bmatrix} + \max\begin{Bmatrix} p(g(x^*),g(a_n)),\\ p(g(x^*),g(b_n)) \end{Bmatrix} \tag{3.14}$$

Letting $n \to \infty$ and by using (3.12) and (3.13), we get $\max\{p(g(x^*),g(x)),p(g(y^*),g(y))\} = 0$.
Hence we have $g(x^*) = g(x), \quad g(y^*) = g(y).$ (3.15)
Next we show that $g(x) = x$ and $g(y) = y$. Let $g(x) = u$ and $g(y) = v$. By the commutativity of F and g and the fact that $g(x) = F(x,y)$ and $g(y) = F(y,x)$, we have

$$\begin{aligned} g(u) = g(g(x)) = g(F(x,y) = F(g(x),g(y)) = F(u,v)\\ g(v) = g(g(y)) = g(F(y,x) = F(g(y),g(x)) = F(v,u) \end{aligned} \tag{3.16}$$

Thus, (u,v) is a coupled coincidence point of F and g. However, according to (3.15), we must have $g(x) = g(u), \quad g(y) = g(v).$
Hence, we have $u = g(u) = F(u,v), \quad v = g(v) = F(v,u).$
That is, the pair (u,v) is the coupled common fixed point of F and g.
Assume that (z,w) is another coupled common fixed point of F and g. But, it follows from (3.15), we get $u = g(u) = g(z) = z, \quad v = g(v) = g(w) = w.$
Hence (u,v) is the unique coupled common fixed point of F and g.

Remark 3.1: *Theorem 3.1, is more general than Theorem 1.2 in view of weak generalized* g_p *-Meir-Keeler contraction. How ever, Thabet Abdeljawad et.al.* [1], *proved Theorem 1.2 by an additional condition, namely, strict* g *-meir-keeler contraction.*

Corollary 3.1: In addition to the hypotheses of Corollary 2.1, assume that for all (x, y) , $(x^*, y^*) \in X^2$, there exists $(a,b) \in X^2$ such that $(F(a,b), F(b,a))$ is comparable to both $(F(x,y), F(y,x))$ and $(F(x^*, y^*), F(y^*, x^*))$ Then, F has a unique coupled fixed point.

Remark 3.2: *Corollary 3.1, is a generalization of Theorem 1.6 of Ali Erduran et.al.* [3]

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