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ON T- CURVATURE TENSOR IN LP-SASAKIAN MANIFOLDS

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ABSTRACT
Some results on the properties of T -flat, quasi-T -flat, ξ-T -flat, φ-T -flat, T -semi-symmetric, φ-T -Ricci recurrent and T -φ-recurrent LP-Sasakian manifolds are obtained. It is also proved that an LP-Sasakian manifold satisfying the condition T $.S = 0$ is an η-Einstein manifold.

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1. INTRODUCTION
The notion of a Lorentzian para-Sasakian manifold was introduced by Matsumoto [3]. Mihai and Rosca defined the same notion independently and they obtained several results on this manifold [5]. LP-Sasakian manifolds have also been studied by Matsumoto and Mihai [4], De *et al*. [2], Shaikh and Biswas [7], Shukla and Shah [8], Tripathi and De [9], Bagewadi *et al.* [1]. On the other hand, Tripathi and Gupta introduced the T -curvature tensor which in particular cases reduces to other some known curvature tensors [10]. Tripathi and Gupta studied T -curvature tensor in K-contact and Sasakian and $(N(k),\xi)$-semi-Riemannian manifolds, respectively [11, 12]. Nagaraja and Somashekhara studied T -curvature tensor in (k,μ)-contact manifolds [6].

The purpose of this paper is to obtain some results on T -curvature tensor in LP-Sasakian manifolds. T -flat, quasi-T -flat, ξ-T -flat, φ-T -flat, T -semi-symmetric, φ-T -Ricci recurrent and T -φ-recurrent LP-Sasakian manifolds are studied. It is also proved that an LP-Sasakian manifold satisfying the condition T $.S = 0$ is an η-Einstein manifold.

We have used the following definitions and notions throughout the paper:
Definition 1.1. An LP-Sasakian manifold (M^{2n+1}, g) is said to be

(1) T -flat if

(1.1) $$\mathrm{T}(X,Y)Z = 0,$$

(2) quasi-T -flat if

(1.2) $$g(\mathrm{T}(X,Y)Z,\varphi W) = 0,$$

(3) ξ-T -flat if

(1.3) $$\mathrm{T}(X,Y)\xi = 0,$$

(4) φ-T -flat if

(1.4) $$g\left(\mathrm{T}\left(\varphi X,\varphi Y\right)\varphi Z,\varphi W\right)=0,$$

for any vector fields X,Y,Z and W.

Definition 1.2. An LP-Sasakian manifold $\left(M^{2n+1},g\right)$ is said to be η-Einstein if its Ricci tensor S is of the form

(1.5) $$S\left(X,Y\right)=\alpha_1 g\left(X,Y\right)+\alpha_2\eta\left(X\right)\eta\left(Y\right)$$

for any vector fields X,Y where α_1,α_2 are smooth functions on the manifold. In particular, if $\alpha_2=0,$ then the manifold is said to be an Einstein manifold.

Definition 1.3. An LP-Sasakian manifold of dimension $\left(2n+1\right)$ is said to be T - semi-symmetric if it satisfies the condition

(1.6) $$R\left(X,Y\right).\mathrm{T}=0.$$

Definition 1.4. An LP-Sasakian manifold $\left(M^{2n+1},g\right)$ is called φ-T -Ricci recurrent if its T -Ricci operator Q_{T} satisfies the condition

(1.7) $$\varphi^2\left(\left(\nabla_W Q_{\mathrm{T}}\right)X\right)=A\left(W\right)Q_{\mathrm{T}}X,$$

where A is a non-zero 1-form.

Definition 1.5. An LP-Sasakian manifold $\left(M^{2n+1},g\right)$ is said to be T -φ-recurrent manifold if there exists a non-zero 1-form A such that

(1.8) $$\varphi^2\left(\left(\nabla_W\mathrm{T}\right)\left(X,Y\right)Z\right)=A\left(W\right)\mathrm{T}\left(X,Y\right)Z,$$

for arbitrary vector fields $X,Y,Z,W,$ where T is a T -curvature tensor. If the 1-form A vanishes, then the manifold reduces to a locally T -φ-symmetric manifold.

2. PRELIMINARIES

A $\left(2n+1\right)$-dimensional differentiable manifold M is called an LP-Sasakian manifold [3], [4] if it admits a (1, 1) tensor field $\varphi,$ a contravariant vector field $\xi,$ a 1-form η and a Lorentzian metric g which satisfy

(2.1) $$\eta\left(\xi\right)=-1,$$

(2.2) $$\varphi^2\left(X\right)=X+\eta\left(X\right)\xi,$$

(2.3) $$g\left(\varphi X,\varphi Y\right)=g\left(X,Y\right)+\eta\left(X\right)\eta\left(Y\right),$$

(2.4) $$\nabla_X\xi=\varphi X,\qquad g\left(X,\xi\right)=\eta\left(X\right),$$

(2.5) $$\left(\nabla_X\varphi\right)\left(Y\right)=g\left(X,Y\right)\xi+\eta\left(Y\right)X+2\eta\left(X\right)\eta\left(Y\right)\xi,$$

where ∇ denotes the covariant differentiation with respect to the Lorentzian metric g.

It can be easily seen that

(2.6) $$\varphi\xi=0,\ \ \eta\left(\varphi X\right)=0,$$

(2.7) $$\text{rank }\varphi=2n.$$

Again, if we put

$$\Phi\left(X,Y\right)=g\left(X,\varphi Y\right),$$

for any vector fields X and $Y,$ then the tensor field $\Phi\left(X,Y\right)$ is a symmetric (0, 2) tensor field [3] and we have

(2.8) $$\Phi\left(X,Y\right)=\Phi\left(Y,X\right)=g\left(X,\varphi Y\right)=g\left(\varphi X,Y\right),$$

for any vector fields X and Y.
Since the 1-form η is closed in an LP-Sasakian manifold, we have [2] [3]

(2.9) $$\begin{cases} (\nabla_X \eta)(Y) = \Phi(X,Y) = g(X,\varphi Y), \\ \Phi(X,\xi) = 0, \end{cases}$$

for any vector fields X and Y.
Further, in a $(2n+1)$-dimensional LP-Sasakian manifold M the following relations hold [2] [4]

(2.10) $$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y),$$

(2.11) $$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X,$$

(2.12) $$R(X,Y)\xi = \eta(Y)X - \eta(X)Y,$$

(2.13) $$R(\xi,X)\xi = X + \eta(X)\xi,$$

(2.14) $$R(X,\xi)\xi = -X - \eta(X)\xi,$$

(2.15) $$S(X,\xi) = 2n\eta(X), \quad S(\xi,\xi) = -2n,$$

(2.16) $$QX = 2nX, \quad Q\xi = 2n\xi,$$

(2.17) $$S(\varphi X,\varphi Y) = S(X,Y) + 2n\eta(X)\eta(Y),$$

for any vector fields $X,Y,Z,$ where $R(X,Y)Z$ is the Riemannian curvature, the Ricci tensor, the Ricci operator of the manifold, respectively.
Also in a $(2n+1)$-dimensional LP-Sasakian manifold M, if $\{e_1, e_2, ..., e_{2n}, \xi\}$ is a local orthonormal basis of vector fields in M, then $\{\varphi e_1, \varphi e_2, ..., \varphi e_{2n}, \xi\}$ is also a local orthonormal basis. It is easy to verify that

(2.18) $$\sum_{i=1}^{2n} g(e_i, e_i) = \sum_{i=1}^{2n} g(\varphi e_i, \varphi e_i) = 2n,$$

(2.19) $$\sum_{i=1}^{2n} g(e_i, Z) S(Y, e_i) = \sum_{i=1}^{2n} g(\varphi e_i, Z) S(Y, \varphi e_i) = S(Y,Z) + 2n\eta(Y)\eta(Z).$$

From (2.15), we have

(2.20) $$\sum_{i=1}^{2n} S(e_i, e_i) = \sum_{i=1}^{2n} S(\varphi e_i, \varphi e_i) = r + 2n,$$

where r is the scalar curvature. In an LP-Sasakian manifold, from (2.11) we also get

(2.21) $$R(\xi,Y,Z,\xi) = -g(\varphi Y, \varphi Z).$$

Consequently,

(2.22) $$\sum_{i=1}^{2n} R(e_i, Y, Z, e_i) = \sum_{i=1}^{2n} R(\varphi e_i, Y, Z, \varphi e_i) = S(Y,Z) + g(\varphi Y, \varphi Z).$$

Similarly, we have

(2.23) $$\sum_{i=1}^{2n} S(\varphi e_i, \varphi Z) g(\varphi Y, \varphi e_i) = S(\varphi Y, \varphi Z),$$

(2.24) $$\sum_{i=1}^{2n} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) = S(\varphi Y, \varphi Z) + g(\varphi Y, \varphi Z).$$

In a $(2n+1)$-dimensional Riemannian manifold, the T -curvature tensor [10] is given by

$$\begin{aligned} \mathrm{T}(X,Y)Z &= a_0R(X,Y)Z + a_1S(Y,Z)X + a_2S(X,Z)Y \\ &+ a_3S(X,Y)Z + a_4g(Y,Z)QX + a_5g(X,Z)QY \\ &+ a_6g(X,Y)QZ + a_7r\big(g(Y,Z)X - g(X,Z)Y\big), \end{aligned} \tag{2.25}$$

where R, S, Q and r are the curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature, respectively; and $a_0, ..., a_7$ are real numbers.

From (2.25) it follows that [6] [12]

$$S_{\mathrm{T}}(X,Y) = (a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6)S(X,Y) + \big(a_4 + (2n-1)a_7\big)rg(X,Y), \tag{2.26}$$

where $S_{\mathrm{T}}(X,Y)$ is the T -Ricci tensor of type (0, 2) and Q_{T} is the T -Ricci operator.

From (2.26), we have

$$Q_{\mathrm{T}}X = (a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6)QX + \big(a_4 + (2n-1)a_7\big)rX. \tag{2.27}$$

In view of (2.16) and (2.27), we obtain

$$Q_{\mathrm{T}}\xi = 2n(a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6)\xi + \big(a_4 + (2n-1)a_7\big)r\xi. \tag{2.28}$$

3. MAIN RESULTS

Now, we prove the following theorems

Theorem 3.1. Let M be a $(2n+1)$-dimensional LP-Sasakian manifold satisfying the condition $\mathrm{T}(X,Y)Z = 0.$

1. If $a_3 \neq 0,$ then M is an η-Einstein manifold. In particular, M becomes an Einstein manifold provided

$$2n(a_1 + a_2 + a_4 + a_5) = 0.$$

2. If $a_3 \neq 0,$ then

$$r = -\frac{2n(a_1 + a_2 + a_3 + a_4 + a_5 + 2na_6)}{a_3}. \tag{3.1}$$

Proof. Let M be a $(2n+1)$-dimensional LP-Sasakian manifold, then for T -flat LP-Sasakian manifold we have $\mathrm{T}(X,Y)Z = 0,$ for any vector fields $X, Y, Z \in \chi(M)$ where $\chi(M)$ is the Lie algebra of vector fields in is M.

Now, from (2.25) and (1.1), we have

$$\begin{aligned} 0 &= a_0g\big(R(X,Y)Z,W\big) + a_1S(Y,Z)g(X,W) + a_2S(X,Z)g(Y,W) \\ &+ a_3S(X,Y)g(Z,W) + a_4g(Y,Z)S(X,W) + a_5g(X,Z)S(Y,W) \\ &+ a_6g(X,Y)S(Z,W) + a_7r\big(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big). \end{aligned} \tag{3.2}$$

In (3.2) putting $Z = W = \xi$ and using (2.1), (2.4), (2.10) and (2.15), we obtain

$$a_3S(X,Y) = -2na_6g(X,Y) = 2n(a_1 + a_2 + a_4 + a_5)\eta(X)\eta(Y). \tag{3.3}$$

This implies that

$$S(X,Y) = A_1g(X,Y) + A_2\eta(X)\eta(Y), \tag{3.4}$$

where

(3.5) $$\begin{cases} A_1 = -\dfrac{2na_6}{a_3}, \\ A_2 = \dfrac{2n\left(a_1 + a_2 + a_4 + a_5\right)}{a_3}. \end{cases}$$

Hence the manifold is η-Einstein.

If $a_3 \neq 0,$ and $2n\left(a_1 + a_2 + a_4 + a_5\right) = 0,$ then (3.4) reduces to

(3.6) $$S\left(X,Y\right) = A_1 g\left(X,Y\right).$$

This implies that the manifold is Einstein.

If $a_3 \neq 0,$ contracting (3.3) we get the result (3.1). Thus the theorem is proved.

Theorem 3.2. Let M be a $\left(2n+1\right)$-dimensional quasi-T -flat LP-Sasakian manifold.

1. If $a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6 \neq 0,$ then the manifold is η-Einstein. In particular, M becomes an Einstein manifold provided

$$a_7 r - a_0 - 2n\left(a_2 + a_3 + a_5 + a_6\right) = 0.$$

2. If $a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6 = 0$ and $2n\left(\left(2n-1\right)a_7 + a_4\right) - a_7 \neq 0,$ then

(3.7) $$r = \frac{2n\left(a_2 + a_3 - 2na_4 + a_5 + a_6\right) - \left(2n-1\right)a_0}{2n\left(\left(2n-1\right)a_7 + a_4 - a_7\right)}.$$

Proof. Let M be a $\left(2n+1\right)$-dimensional LP-Sasakian manifold. For quasi- T -flat LP-Sasakian manifold, from (2.25) and (1.2) we have

(3.8) $$\begin{aligned} 0 &= a_0 g\left(R\left(X,Y\right)Z,\varphi W\right) + a_1 S\left(Y,Z\right)g\left(X,\varphi W\right) + a_2 S\left(X,Z\right)g\left(Y,\varphi W\right) \\ &+ a_3 S\left(X,Y\right)g\left(Z,\varphi W\right) + a_4 g\left(Y,Z\right)S\left(X,\varphi W\right) + a_5 g\left(X,Z\right)S\left(Y,\varphi W\right) \\ &+ a_6 g\left(X,Y\right)S\left(Z,\varphi W\right) + a_7 r\left(g\left(Y,Z\right)g\left(X,\varphi W\right) - g\left(X,Z\right)g\left(Y,\varphi W\right)\right). \end{aligned}$$

Replacing X by φX in (3.8), we get

(3.9) $$\begin{aligned} 0 &= a_0 g\left(R\left(\varphi X,Y\right)Z,\varphi W\right) + a_1 S\left(Y,Z\right)g\left(\varphi X,\varphi W\right) + a_2 S\left(\varphi X,Z\right)g\left(Y,\varphi W\right) \\ &+ a_3 S\left(\varphi X,Y\right)g\left(Z,\varphi W\right) + a_4 g\left(Y,Z\right)S\left(\varphi X,\varphi W\right) + a_5 g\left(\varphi X,Z\right)S\left(Y,\varphi W\right) \\ &+ a_6 g\left(\varphi X,Y\right)S\left(Z,\varphi W\right) + a_7 r(g\left(Y,Z\right)g\left(\varphi X,\varphi W\right) - g\left(\varphi X,Z\right)g\left(Y,\varphi W\right)). \end{aligned}$$

If $\left\{e_1, e_2, ..., e_{2n}, \xi\right\}$ is a local orthonormal basis of vector fields in M, then $\left\{\varphi e_1, \varphi e_2, ..., \varphi e_{2n}, \xi\right\}$ is also a local orthonormal basis. Now, from (3.9) we get

(3.10) $$\begin{aligned} 0 &= a_0 \sum_{i=1}^{2n} g\left(R\left(\varphi e_i, Y\right)Z, \varphi e_i\right) + a_1 \sum_{i=1}^{2n} S\left(Y,Z\right)g\left(\varphi e_i, \varphi e_i\right) \\ &+ a_2 \sum_{i=1}^{2n} S\left(\varphi e_i, Z\right)g\left(Y, \varphi e_i\right) + a_3 \sum_{i=1}^{2n} S\left(\varphi e_i, Y\right)g\left(Z, \varphi e_i\right) \\ &+ a_4 \sum_{i=1}^{2n} g\left(Y,Z\right)S\left(\varphi e_i, \varphi e_i\right) + a_5 \sum_{i=1}^{2n} g\left(\varphi e_i, Z\right)S\left(Y, \varphi e_i\right) \end{aligned}$$

$$+a_6\sum_{i=1}^{2n} g(\varphi e_i, Y) S(Z, \varphi e_i) + a_7 r(\sum_{i=1}^{2n} g(Y,Z) g(\varphi e_i, \varphi e_i) - \sum_{i=1}^{2n} g(\varphi e_i, Z) g(Y, \varphi e_i)).$$

Using (2.18) – (2.22) and (2.3) in (3.10), we obtain

$$\begin{aligned}&(a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6) S(Y,Z)\\ &= (a_7 r - a_0 - a_4 r - 2n(a_4 + a_7 r)) g(Y,Z)\\ &+ (a_7 r - a_0 - 2n(a_2 + a_3 + a_5 + a_6)) \eta(Y)\eta(Z).\end{aligned} \tag{3.11}$$

This implies that

$$S(Y,Z) = B_1 g(Y,Z) + B_2 \eta(Y)\eta(Z), \qquad a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6 \neq 0, \tag{3.12}$$

where

$$B_1 = \frac{a_7 r - a_0 - a_4 r - 2n(a_4 + a_7 r)}{a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6} \tag{3.13}$$

and

$$B_2 = \frac{a_7 r - a_0 - 2n(a_2 + a_3 + a_5 + a_6)}{a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6}. \tag{3.14}$$

Hence the manifold is η-Einstein.

If $a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6 \neq 0$ and $a_7 r - a_0 - 2n(a_2 + a_3 + a_5 + a_6) = 0,$ then (3.12) reduces to

$$S(Y,Z) = B_1 g(Y,Z). \tag{3.15}$$

This implies that the manifold is Einstein.

If $a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6 = 0,$ contracting (3.11) we get (3.7). This completes the proof of the theorem.

Theorem 3.3. Let M be a $(2n+1)$-dimensional ξ-T -flat LP-Sasakian manifold.

1. If $a_4 \neq 0,$ then the manifold is η-Einstein. In particular, M becomes an Einstein manifold provided

$$2n(a_2 + a_3 + a_5 + a_6) - a_0 - a_7 r = 0.$$

2. If $a_4 = 0$ and $(2n-1)a_7 \neq 0,$ then

$$r = -\frac{2n(a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6)}{(2n-1)a_7}. \tag{3.16}$$

Proof. Let us consider an LP-Sasakian manifold of dimension $(2n+1)$ which is ξ-T -flat. From (2.25) and (1.3), we have

$$\begin{aligned}0 &= a_0 R(X,Y)\xi + a_1 S(Y,\xi) X + a_2 S(X,\xi) Y\\ &+ a_3 S(X,Y)\xi + a_4 g(Y,\xi) QX + a_5 g(X,\xi) QY\\ &+ a_6 g(X,Y) Q\xi + a_7 r(g(Y,\xi) X - g(X,\xi) Y).\end{aligned} \tag{3.17}$$

Taking inner product on both sides of (3.17) by W and using (2.4) and (2.15), we get

$$\begin{aligned}0 &= a_0 R(X,Y,\xi,W) + 2na_1 g(X,W)\eta(Y) + 2na_2 g(Y,W)\eta(X) \\ &+ a_3 S(X,Y)\eta(W) + a_4 S(X,W)\eta(Y) + a_5 S(Y,W)\eta(X) \\ &+ 2na_6 g(X,Y)\eta(W) + a_7 r\big(g(X,W)\eta(Y) - g(Y,W)\eta(X)\big).\end{aligned} \tag{3.18}$$

Putting $Y = \xi$ in (3.18) and using (2.1), (2.4), (2.14) and (2.15), we obtain

$$\begin{aligned}a_4 S(X,W) &= -(a_0 + 2na_1 + a_7 r) g(X,W) \\ &\quad + \big(2n(a_2 + a_3 + a_5 + a_6) - a_0 - a_7 r\big)\eta(X)\eta(W).\end{aligned} \tag{3.19}$$

This implies that

$$S(X,W) = C_1 g(X,W) + C_2 \eta(X)\eta(W), \qquad a_4 \neq 0, \tag{3.20}$$

where

$$C_1 = -\frac{a_0 + 2na_1 + a_7 r}{a_4} \tag{3.21}$$

and

$$C_2 = \frac{2n(a_2 + a_3 + a_5 + a_6) - a_0 - a_7 r}{a_4}. \tag{3.22}$$

Thus the manifold M is η-Einstein.

If $a_4 \neq 0$ and $2n(a_2 + a_3 + a_5 + a_6) - a_0 - a_7 r = 0,$ then (3.20) yields

$$S(X,W) = C_1 g(X,W). \tag{3.23}$$

Hence the manifold is Einstein.

If $a_4 = 0,$ then contracting (3.19) we obtain (3.16). This proves the result.

Theorem 3.4. If a $(2n+1)$-dimensional LP-Sasakian manifold is φ-T -flat such that $a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6 \neq 0,$ then the manifold is η-Einstein.

Proof. Let M be a $(2n+1)$-dimensional LP-Sasakian manifold. For φ-T -flat LP-Sasakian manifold the condition (3.4) holds.

Now, from (2.25) and (1.4), we get

$$\begin{aligned}0 &= a_0 g\big(R(\varphi X,\varphi Y)\varphi Z,\varphi W\big) + a_1 S(\varphi Y,\varphi Z) g(\varphi X,\varphi W) \\ &+ a_2 S(\varphi X,\varphi Z) g(\varphi Y,\varphi W) + a_3 S(\varphi X,\varphi Y) g(\varphi Z,\varphi W) \\ &+ a_4 g(\varphi Y,\varphi Z) S(\varphi X,\varphi W) + a_5 g(\varphi X,\varphi Z) S(\varphi Y,\varphi W) \\ &+ a_6 g(\varphi X,\varphi Y) S(\varphi Z,\varphi W) + a_7 r(g(\varphi Y,\varphi Z) g(\varphi X,\varphi W) \\ &- g(\varphi X,\varphi Z) g(\varphi Y,\varphi W)).\end{aligned} \tag{3.24}$$

If $\{e_1, e_2, ..., e_{2n}, \xi\}$ is a local orthonormal basis of vector fields in M, then $\{\varphi e_1, \varphi e_2, ..., \varphi e_{2n}, \xi\}$ is also a local orthonormal basis. Now, from (3.24) we get

$$
\begin{aligned}
0 = {} & a_0 \sum_{i=1}^{2n} g\left(R\left(\varphi e_i, \varphi Y\right)\varphi Z, \varphi e_i\right) + a_1 \sum_{i=1}^{2n} S\left(\varphi Y, \varphi Z\right) g\left(\varphi e_i, \varphi e_i\right) \\
& + a_2 \sum_{i=1}^{2n} S\left(\varphi e_i, \varphi Z\right) g\left(\varphi Y, \varphi e_i\right) + a_3 \sum_{i=1}^{2n} S\left(\varphi e_i, \varphi Y\right) g\left(\varphi Z, \varphi e_i\right) \\
& + a_4 \sum_{i=1}^{2n} g\left(\varphi Y, \varphi Z\right) S\left(\varphi e_i, \varphi e_i\right) + a_5 \sum_{i=1}^{2n} g\left(\varphi e_i, \varphi Z\right) S\left(\varphi Y, \varphi e_i\right) \\
& + a_6 \sum_{i=1}^{2n} g\left(\varphi e_i, \varphi Y\right) S\left(\varphi Z, \varphi e_i\right) + a_7 r \sum_{i=1}^{2n} (g\left(\varphi Y, \varphi Z\right) S\left(\varphi e_i, \varphi e_i\right) \\
& - g\left(\varphi e_i, \varphi Z\right) g\left(\varphi Y, \varphi e_i\right)).
\end{aligned} \tag{3.25}
$$

Using (2.18), (2.20), (2.23) and (2.24) in (3.25), we get

$$
\begin{aligned}
0 = {} & \left(a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6\right) S\left(\varphi Y, \varphi Z\right) \\
& + \left(a_0 + \left(2n + r\right) a_4 + \left(2n - 1\right) a_7 r\right) g\left(\varphi Y, \varphi Z\right).
\end{aligned} \tag{3.26}
$$

In view of (2.3), (2.17) and (3.26), we obtain

$$
S\left(Y, Z\right) = D_1 g\left(Y, Z\right) + D_2 \eta\left(Y\right)\eta\left(Z\right), \qquad a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6 \neq 0, \tag{3.27}
$$

where

$$
D_1 = -\frac{a_0 + 2na_4 + \left(a_4 + \left(2n-1\right)a_7\right) r}{a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6} \tag{3.28}
$$

and

$$
D_2 = -\frac{\left(2n+1\right)a_0 + 2n\left(2na_1 + a_2 + a_3 + a_4 + a_5 + a_6\right) + \left(a_4 + \left(2n-1\right)a_7\right) r}{a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6}. \tag{3.29}
$$

Hence the manifold M is η-Einstein. This proves the theorem.

Theorem 3.5. A T -semi-symmetric LP-Sasakian manifold $\left(M^{2n+1}, g\right)$ is an η-Einstein manifold provided $a_0 + a_5 + a_6 \neq 0$.

Proof. Let M be a $\left(2n+1\right)$-dimensional LP-Sasakian manifold satisfying the condition (1.6).

Now,

$$
\begin{aligned}
\left(R\left(X, Y\right).\mathrm{T}\right)\left(U, V\right) Z = {} & R\left(X, Y\right)\mathrm{T}\left(U, V\right) Z - \mathrm{T}\left(R\left(X, Y\right)U, V\right) Z \\
& - \mathrm{T}\left(U, R\left(X, Y\right)V\right) Z - \mathrm{T}\left(U, V\right) R\left(X, Y\right) Z.
\end{aligned} \tag{3.30}
$$

From (1.6) and (3.30), we have

$$
\begin{aligned}
0 = {} & g\left(R\left(\xi, Y\right)\mathrm{T}\left(U, V\right) Z, \xi\right) - g\left(\mathrm{T}\left(R\left(\xi, Y\right)U, V\right) Z, \xi\right) \\
& - g\left(\mathrm{T}\left(U, R\left(\xi, Y\right)V\right) Z, \xi\right) - g\left(\mathrm{T}\left(U, V\right) R\left(\xi, Y\right) Z, \xi\right).
\end{aligned} \tag{3.31}
$$

Putting $Y = U$ in (3.31) and using (2.1) and (2.11), we get

$$
\begin{aligned}
0 = {} & g\left(\mathrm{T}\left(U, V\right) Z, U\right) + g\left(U, U\right)\eta\left(\mathrm{T}\left(\xi, V\right) Z\right) + g\left(U, V\right)\eta\left(\mathrm{T}\left(U, \xi\right) Z\right) \\
& - \eta\left(V\right)\eta\left(\mathrm{T}\left(U, U\right) Z\right) + g\left(U, Z\right)\eta\left(\mathrm{T}\left(U, V\right)\xi\right) - \eta\left(Z\right)\eta\left(\mathrm{T}\left(U, V\right) U\right).
\end{aligned} \tag{3.32}
$$

Let $\left\{e_1, e_2, ..., e_{2n}, \xi\right\}$ be an orthonormal basis of vector fields in M, then (3.32) can be written as

(3.33)
$$\begin{aligned}0 &= \sum_{i=1}^{2n} g\left(\mathrm{T}\left(e_i,V\right)Z,e_i\right)+\sum_{i=1}^{2n} g\left(e_i,e_i\right)\eta\left(\mathrm{T}\left(\xi,V\right)Z\right)\\ &+\sum_{i=1}^{2n} g\left(e_i,V\right)\eta\left(\mathrm{T}\left(e_i,\xi\right)Z\right)-\sum_{i=1}^{2n}\eta\left(V\right)\eta\left(\mathrm{T}\left(e_i,e_i\right)Z\right)\\ &+\sum_{i=1}^{2n} g\left(e_i,Z\right)\eta\left(\mathrm{T}\left(e_i,V\right)\xi\right)-\sum_{i=1}^{2n}\eta\left(Z\right)\eta\left(\mathrm{T}\left(e_i,V\right)e_i\right).\end{aligned}$$

In view of (2.18) – (2.22) and (2.25), (3.33) reduces to

(3.34) $$S\left(V,Z\right)=E_1 g\left(V,Z\right)+E_2\eta\left(V\right)\eta\left(Z\right), \qquad a_0+a_5+a_6\neq 0,$$

where

(3.35) $$E_1=\frac{2\left(n-1\right)a_0+2n\left(\left(2n-1\right)a_4+a_5+a_6\right)-a_4 r}{a_0+a_5+a_6}$$

and

(3.36) $$E_2=-\frac{a_0+2n\left(2a_1+2na_2+2na_3+a_5+a_6\right)-\left(a_2+a_3\right)r}{a_0+a_5+a_6}.$$

The relation (3.34) implies that the manifold M is η-Einstein. This completes the proof of the theorem.

Theorem 3.6. A φ-T-Ricci recurrent LP-Sasakian manifold $\left(M^{2n+1},g\right)$ is an Einstein manifold.

Proof. Let us consider an LP-Sasakian manifold $\left(M^{2n+1},g\right)$. Now, using (2.2) in (1.7) we get

(3.37) $$\left(\nabla_W Q_{\mathrm{T}}\right)X+\eta\left(\left(\nabla_W Q_{\mathrm{T}}\right)X\right)\xi=A\left(W\right)Q_{\mathrm{T}}X.$$

Putting $X=\xi$ and taking inner product on both sides of (3.37) by $Z,$ we get

(3.38)
$$\begin{aligned}A\left(W\right)g\left(Q_{\mathrm{T}}\xi,Z\right) &= g\left(\nabla_W Q_{\mathrm{T}}\xi,Z\right)-g\left(Q_{\mathrm{T}}\left(\nabla_W\xi\right),Z\right)\\ &+\eta\left(\left(\nabla_W Q_{\mathrm{T}}\right)\xi\right)\eta\left(Z\right).\end{aligned}$$

Replacing Z by φZ and using (2.4), (2.6), (2.26) and (2.28) in (3.38), we obtain

(3.39)
$$\begin{aligned}S_{\mathrm{T}}\left(\varphi W,\varphi Z\right) &= \{2n\left(a_0+2na_1+a_2+a_3+a_5+a_6\right)\\ &+\left(a_4+\left(2n-1\right)a_7\right)r\}g\left(\varphi W,\varphi Z\right).\end{aligned}$$

By virtue of (2.3), (2.17), (2.26) and (3.39), we get

(3.40) $$S\left(W,Z\right)=2ng\left(W,Z\right).$$

Hence the manifold M is Einstein. This proves the result.

Theorem 3.7. A T-φ-recurrent LP-Sasakian manifold of dimension $\left(2n+1\right)$ is an Einstein manifold, provided $a_0+2na_1+a_2+a_5+a_6\neq 0.$

Proof. Let us consider a T-φ-recurrent LP-Sasakian manifold $\left(M^{2n+1},g\right)$. Then by virtue of (1.8) and (2.1), we have

(3.41) $$\left(\nabla_W \mathrm{T}\right)\left(X,Y\right)Z+\eta\left(\left(\nabla_W \mathrm{T}\right)\left(X,Y\right)Z\right)\xi=A\left(W\right)\mathrm{T}\left(X,Y\right)Z,$$

from which it follows that

(3.42) $$g\left(\left(\nabla_W \mathrm{T}\right)\left(X,Y\right)Z,U\right)+\eta\left(\left(\nabla_W \mathrm{T}\right)\left(X,Y\right)Z\right)\eta\left(U\right)=A\left(W\right)g\left(\mathrm{T}\left(X,Y\right)Z,U\right).$$

Let $\{e_i\}$, $i=1,2,...,2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X=U=e_i$ in (3.42) and taking summation over i, $1\le i\le 2n+1$, we get

(3.43) $$\sum_{i=1}^{2n+1} g\left(\left(\nabla_W \mathrm{T}\right)\left(e_i,Y\right)Z,e_i\right)+\sum_{i=1}^{2n+1}\eta\left(\left(\nabla_W \mathrm{T}\right)\left(e_i,Y\right)Z\right)\eta\left(e_i\right) = \sum_{i=1}^{2n+1} A(W)\, g\left(\mathrm{T}\left(e_i,Y\right)Z,e_i\right).$$

In view of (2.15), (2.16), (2.25) and (3.43), we obtain

(3.44) $$\begin{aligned} &A(W)\left[\left(a_0+(2n+1)a_1+a_3+a_5+a_6\right)S(Y,Z)+\left(a_4+2na_7\right)rg(Y,Z)\right] \\ &=\left(a_0+2na_1+a_2+a_3+a_5+a_6\right)\left(\nabla_W S\right)(Y,Z)+(2n-1)a_7 dr(W)\,g(Y,Z) \\ &\quad+a_2\left(\nabla_W S\right)(Z,\xi)\eta(Y)+a_3\left(\nabla_W S\right)(Y,\xi)\eta(Z)-a_7 dr(W)\eta(Y)\eta(Z). \end{aligned}$$

Putting $Z=\xi$ and using (2.1), (2.4) and (2.15) in (3.44), we get

(3.45) $$\begin{aligned} &A(W)\left[2n\left(a_0+(2n+1)a_1+a_3+a_5+a_6\right)+\left(a_4+2na_7\right)r\right]\eta(Y) \\ &=\left(a_0+2na_1+a_2+a_5+a_6\right)\left(\nabla_W S\right)(Y,\xi) \\ &\quad+2na_7 dr(W)\eta(Y)+a_2\left(\nabla_W S\right)(\xi,\xi)\eta(Y). \end{aligned}$$

Now, we have

$$\left(\nabla_W S\right)(Y,\xi)=\nabla_W S\left((Y,\xi)\right)-S\left(\nabla_W Y,\xi\right)-S\left(Y,\nabla_W \xi\right).$$

Using (2.4), (2.9) and (2.15) in above relation we obtain

(3.46) $$\left(\nabla_W S\right)(Y,\xi)=2ng(Y,\varphi W)-S(Y,\varphi W)$$

and

(3.47) $$\left(\nabla_W S\right)(\xi,\xi)=0.$$

Replacing Y by φY in (3.45) and using (3.46), (3.47), (2.3), (2.6) and (2.17), we get

(3.48) $$S(Y,W)=2ng(Y,W), \qquad a_0+2na_1+a_2+a_5+a_6\neq 0.$$

This implies that the manifold is Einstein. This proves the theorem.

Theorem 3.8. A $(2n+1)$-dimensional LP-Sasakian manifold satisfying the condition $\mathrm{T}.S=0$ is an η-Einstein manifold, provided $a_3\neq 0.$

Proof. Let $\left(M^{2n+1},g\right)$ be an LP-Sasakian manifold. Suppose $\left(M^{2n+1},g\right)$ satisfies the condition $\mathrm{T}.S=0;$ then we have

(3.49) $$\left(\mathrm{T}(U,V).S\right)(Y,\xi)=0,$$

from which we get

(3.50) $$S\left(\mathrm{T}(U,V)Y,\xi\right)+S\left(Y,\mathrm{T}(U,V)\xi\right)=0.$$

Taking $Y=\xi$ in (3.50), we obtain

(3.51) $$S\left(\mathrm{T}(U,V)\xi,\xi\right)=0.$$

Using (2.15) and (2.25) in (3.51), we get

(3.52) $$S(U,V)=F_1 g(U,V)+F_2\eta(U)\eta(V), \qquad a_3\neq 0,$$

where

(3.53) $$F_1=-\frac{2na_6}{a_3}$$

and

(3.54) $$F_2 = \frac{2n\left(a_1 + a_2 + a_4 + a_5\right)}{a_3}.$$

Thus from the relation (3.52) the manifold is η-Einstein. This proves the result.

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