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A GENERALIZATION OF CONTRACTION PRINCIPLE ON PARTIAL METRIC SPACES

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ABSTRACT

In this paper we have proved fixed point theorem using continuous and monotonically non-decreasing mapping $\phi,\psi:[0,\infty)\to[0,\infty)$ with $\phi(0)=\psi(0)=0$.

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1 INTRODUCTION

In 1992, Matthews [1, 2] introduced the notion of a partial metric space which is a generalization of usual metric spaces in which d(x; x) is not necessarily zero. In this paper we have proved _xed point theorem using continuous and monotonically non-decreasing mapping $\phi,\psi:[0,\infty)\to[0,\infty)$ with $\phi(0)=\psi(0)=0$.

2 PRELIMINARY NOTES

Definition 2.1 Let X be a non-empty set. Suppose the mapping $\rho: X\times X:[0,\infty)$ is said to be a partial metric on X, if for any $x,y,z\in X$ the following conditions hold true:

(p1) $\rho(x,y)=\rho(y,x)$ (symmetry),

(p2) If $\rho(x,x)=\rho(x,y)=\rho(y,y)$, then x = y (equality),

(p3) $\rho(x,x)\le\rho(x,y)$ (small self distances),

(p4) $\rho(x,z)\le\rho(x,y)+\rho(y,z)-\rho(y,y)$ (triangularity),

then (X,ρ) is called a partial metric space (see, e.g. [1, 2]).

Notice that for a partial metric ρ on X, the function $d_\rho: X\times X\to[0,\infty)$ given by

$$d_\rho(x,y)=2\rho(x,y)-\rho(x,x)-\rho(y,y)$$

is a usual metric on X. Observe that each partial metric ρ on X generates a T_0 topology T_ρ on X with a base of the family of open ρ-balls $\{B_\rho(x,\varepsilon): x\in X, \varepsilon>0\}$, where $B_\rho(x,\varepsilon)=\{y\in X:\rho(x,y)<\rho(x,x)+\varepsilon\}$ for all $x\in X,\varepsilon>0$. Similarly, closed ρ -ball is defined as $B_\rho[x,\varepsilon]=\{y\in X:\rho(x,y)\le\rho(x,x)+\varepsilon\}$.

Definition 2.2 [1, 2]

(i) A sequence $\{x_n\}$ in a partial metric space (X,ρ) converges to $x\in X$ if and only if $\rho(x,y)=\lim_{n\to\infty}\rho(x,x_n)$,

(ii) A sequence $\{x_n\}$ in a partial metric space (X,ρ) is called Cauchy if and only if $\lim_{n,m\to\infty}\rho(x_n,x_m)$, exists and is finite,

(iii) A partial metric space (X,ρ) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to T_ρ, to a point $x\in X$ such that $\rho(x,y)=\lim_{n,m\to\infty}\rho(x_n,x_m)$,

(iv) A mapping $f: X\to X$ is said to be continuous at $x_0\in X$ if for every $\varepsilon>0$, there exist $\delta>0$ such that $f(B(x_0,\delta))\subset f(B(x_0,\varepsilon))$.

Lemma 2.3 [1, 2]

(i) A sequence $\{x_n\}$ is Cauchy in a partial metric space (X,ρ) if and only if sequence $\{x_n\}$ is Cauchy in a metric space (X,d_ρ),

(ii) A partial metric space (X,ρ) is complete if and only if a metric space (X,d_ρ) is complete.

Moreover

$$\lim_{n\to\infty} d_\rho(x,x_n)=0 \Leftrightarrow \rho(x,y)=\lim_{n\to\infty}\rho(x,x_n)=\lim_{n,m\to\infty}\rho(x_n,x_m).$$

Main results

Theorem 2.4 Let (X,ρ) be complete partial metric space and let $T: X\to X$ is a self mapping satisfying the inequality

$$\psi[\rho(Tx,Ty)]\le\psi[\rho(x,y)]-\phi[\rho(x,y)], \tag{1}$$

where $\psi,\phi:[0,\infty)\to[0,\infty)$ both are continuous and monotonically non-decreasing functions with $\phi(0)=\psi(0)=0$ if and only if t = 0. Then T having fixed point.

Proof: For any $x_0\in X$, we construct a sequence $\{x_n\}$ by $x_n=Tx_{n-1},\ n=1,2,\cdots$. Substituting $x_n=x_{n-1}$ and $y=x_n$. So the equation (1) becomes

$$\psi[\rho(Tx_n,Tx_{n+1})]\le\psi[\rho(x_{n-1},x_n)]-\phi[\rho(x_{n-1},x_n)], \tag{2}$$

which implies

$$\rho(x_n,x_{n+1})\le\rho(x_{n-1},x_n) \qquad (\because \text{ is monotonic function}).$$

It follows that the sequence $\{\rho(x_n,x_{n+1})\}$ is monotone decreasing and there exist $r\ge 0$ such that

$$\rho(x_n,x_{n+1})\to r \text{ as } n\to\infty.$$

Letting $n\to\infty$ equation (2), becomes

$$\psi(r)\le\psi(r)-\phi(r),$$

which is a contradiction. Hence

$$\rho(x_n,x_{n+1})\to r \text{ as } n\to\infty. \tag{3}$$

To prove sequence $\{x_n\}$ is Cauchy. If suppose $\{x_n\}$ is not Cauchy sequence. Then there exist $\varepsilon>0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with n(k) > m(k) > k such that

$$\rho(x_{m(k)},x_{n(k)})\ge\varepsilon. \tag{4}$$

Further, corresponds to m(k), we can choose n(k) in such that it is the smallest integer with n(k) > m(k) satisfies (4). Then

$$\rho(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{5}$$

Then we have

$$\begin{aligned}\varepsilon &\leq \rho(x_{m(k)}, x_{n(k)}) \\ &\leq \rho(x_{m(k)}, x_{n(k)-1}) + \rho(x_{n(k)-1}, x_{n(k)}) - \leq \rho(x_{n(k)-1}, x_{n(k)-1}) \\ &\leq \varepsilon + \rho(x_{n(k)-1}, x_{n(k)}).\end{aligned}$$

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} \rho(x_{m(k)}, x_{n(k)}) = \varepsilon. \tag{6}$$

Again

$$\rho(x_{m(k)}, x_{n(k)}) \leq \rho(x_{n(k)}, x_{n(k)-1}) + \rho(x_{n(k)-1}, x_{n(k)}) - \rho(x_{n(k)-1}, x_{n(k)-1})$$

$$\rho(x_{n(k)-1}, x_{m(k)-1}) \leq \rho(x_{n(k)-1}, x_{m(k)}) + \rho(x_{m(k)}, x_{m(k)-1}) - \rho(x_{m(k)}, x_{m(k)}).$$

from (5) and (6), we get

$$\lim_{k\to\infty} \rho(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon \tag{7}$$

from (1) and (4), we get

$$\begin{aligned}\psi(\varepsilon) &\leq \psi(\rho(x_{m(k)}, x_{n(k)})) \\ &\leq \psi(\rho(x_{m(k)-1}, x_{n(k)-1})) - \phi(\rho(x_{n(k)-1}, x_{n(k)-1}))\end{aligned}$$

from (6) and (7), we get

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon)$$

which is contradiction if $\varepsilon > 0$. This shows that the sequence $\{x_n\}$ is Cauchy and hence convergent in the complete metric space X say z. So

$$\psi(\rho(x_n, Tz)) \leq \psi(\rho(x_{n-1}, z)) - \phi(\rho(x_{n-1}, z))$$

As $n \to \infty$, and continuity of ψ and ϕ, we have

$$\begin{aligned}\psi(\rho(z, Tz)) &\leq \psi(0) - \phi(0), \\ \rho(z, Tz) &= 0, \\ z &= Tz.\end{aligned}$$

For uniqueness suppose z_1 and z_2 are fixed points of T from (1), we can get

$$\psi(\rho(Tz_1, Tz_2)) \leq \psi(\rho(z_1, z_2)) - \phi(\rho(z_1, z_2))$$

$$\psi(\rho(z_1, z_2)) \leq \psi(\rho(z_1, z_2)) - \phi(\rho(z_1, z_2)).$$

Therefore

$\phi(\rho(z_1, z_2)) = 0$ gives $\rho(z_1, z_2) = 0$, that is $z_1 = z_2$. This proves the uniqueness of the theorem.

Example 2.5 Let $X = [0, \infty)$ and $\rho(x, y) = \max(x, y)$, then (X, ρ) is complete partial metric space. Let $T : X \to X$ such that $Tx = \frac{x}{2}$ and $\psi, \phi : [0, \infty) \to [0, \infty)$ such that $\psi(t) = \frac{t}{2}$ and $\psi(t) = \frac{t}{4}$, holds equation (1)

$$\psi[\rho(Tx, Ty)] \le \psi[\rho(x, y)] - \phi[\rho(x, y)]$$

therefore x = 0 is only fixed point.

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