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ON THE INVERSE LAPLACE TRANSFORM

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ABSTRACT
In this work we show that the Tuan-Duc formula used to invert the Laplace transform is equivalent to the expression given by Post-Widder.

Keywords: Laplace transform; Bromwich formula; Post-Widder and Tuan-Duc equations

INTRODUCTION

The Laplace transform of $f(x)$ is defined as [1]:

$$\mathrm{L}\left[f(x)\right] = F(s) = \int_0^\infty e^{-sx} f(x)\, dx, \tag{1}$$

and the inverse problem is to determine $f(t)$ for a given $F(s)$. Bromwich [2, 3] gave the expression:

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st} f(s)\, ds, \tag{2}$$

where the integration is performed in the complex plane along the straight line $x = \sigma$.
But due to the interest to get the inverse Laplace transform without complex variable, Post [4] and Widder [5, 7] found the following formula in a real variable:

$$f(t) = \lim_{n\to\infty} \frac{(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1}\left[\frac{d^n F}{ds^n}\right]_{s=\frac{n}{t}} \tag{3}$$

In Sec. 2 it is given an alternative form for (3) to obtain a procedure for the recent expression of Tuan-Duc [6]:

$$f(t) = \lim_{n\to\infty} \prod_{k=1}^{n}\left(1+\frac{t}{k}\frac{d}{dt}\right)\left[\frac{n}{t}F\left(\frac{n}{t}\right)\right] \tag{4}$$

and there it will be clear that (4) is another manner to write (3).

THE POST-WIDDER AND TUAN-DUC FORMULAE

First, we know that [1]:

$$\mathrm{L}\left[x^n\right] = \int_0^\infty e^{-sx} x^n\, dx = \frac{n!}{s^{n+1}}, \tag{5}$$

where in $s = \frac{n}{t}$ imlplies:

$$\frac{n^{n+1}}{n!t^{n+1}} \int_0^\infty e^{-\frac{n}{t}x} x^n dx = 1 \tag{6}$$

On the other hand, the binomial theorem of Newton gives:

$$(x-t)^r = \sum_{k=0}^{r} \binom{r}{k} x^k (-t)^{r-k}, \tag{7}$$

such that for r $\geq$ 1:

$$\frac{n^{n+1}}{n!\ t^{n+1}} \int_0^\infty e^{-\frac{n}{t}x}\ x^n\ (x-t)^r\, dx \tag{8}$$

$$= \frac{n^{n+1}}{n!\ t^{n+1}} \int_0^\infty e^{-\frac{n}{t}x}\ x^n \sum_{k=0}^{r} \binom{r}{k} x^k (-t)^{r-k}\, dx,$$

where we employ the gamma function

$$\Gamma(n+k+1) = \int_0^\infty e^u u^{n+k} du = (n+k)!,$$

and our integral adopts the form:

$$t^r \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} \frac{(n+1)(n+2)\ldots(n+k)}{n^k}$$

$$\underset{n\to\infty}{\to} t^r \sum_{k=0}^{r} \binom{r}{k} (1)^k\ (-1)^{r-k} = t^r\ (1-1)^r = 0$$

The expansion of $f(x)$ in Taylor series around $x = t$:

$$f(x) = f(t) + \sum_{r=0}^{\infty} \frac{f^{(r)}(t)}{r!} (x-t)^r \tag{9}$$

together with (6) and (8), leads to the relation:

$$f(t) = \lim_{n\to\infty} \frac{n^{n+1}}{n!\ t^{n+1}} \int_0^\infty e^{-\frac{n}{t}x}\ x^n\ f(x)\, dx \tag{10}$$

Further, from (1) it is immediate that:

$$\left(t^2 \frac{d}{d\,t}\right) F\left(\frac{n}{t}\right) = n \int_0^\infty e^{-\frac{n}{t}x}\ x^n\ f(x)\, dx,$$

and after successive applications of the operator $\left(t^2 \frac{d}{dt} \right)$ we get the identity:

$$\frac{1}{(n-1)!\ t^{n+1}} \left(t^2 \frac{d}{d\,t} \right)^n F\left(\frac{n}{t}\right) = \frac{n^{n+1}}{n!\ t^{n+1}} \int_0^\infty e^{-\frac{n}{t}x} x^n \ f(x)\, dx, \tag{11}$$

which by substitution into (10) gives the inversion formula:

$$f(t) = \lim_{n\to\infty} \frac{1}{(n-1)!\ t^{n+1}} \left(t^2 \frac{d}{d\,t} \right)^n F\left(\frac{n}{t}\right) , \tag{12}$$

allowing the construction of $f(t)$ from $F(s)$.
It is simple to prove that (12) leads to the expression (3) deduced by Post [4]–Widder [5, 7]; in fact, with $s = \frac{n}{t}$:

$$t^2 \frac{d}{dt} F\left(\frac{n}{t}\right) = t^2 \frac{d\,s}{d\,t} \left[\frac{dF}{ds}\right]_{s=\frac{n}{t}} = -\,n \left[\frac{dF}{ds}\right]_{s=\frac{n}{t}} ,$$

so

$$\left(t^2 \frac{d}{d\,t} \right)^n F\left(\frac{n}{t}\right) = (-1)^n\ n^n \left[\frac{d^n\ F}{d\ s^n}\right]_{s=\frac{n}{t}} , \tag{13}$$

which in (12) implies (3). That is, the Post-Widder relation is the simplified form of (12).

From (1), it can be shown that:

$$\left(1 + t\frac{d}{d\,t}\right)\left[\frac{n}{t}\ F\left(\frac{n}{t}\right)\right] = \frac{n^2}{t^2} \int_0^\infty e^{-\frac{n}{t}x}\ x\ \ f(x)\, dx,$$

then

$$\left(1 + \frac{t}{2}\frac{d}{dt}\right)\left(1 + t\frac{d}{dt}\right)\left[\frac{n}{t}\ F\left(\frac{n}{t}\right)\right] = \frac{n^3}{2\,t^3} \int_0^\infty e^{-\frac{n}{t}x}\ x^2\ \ f(x)\, dx, \text{ etc.}$$

and the identity:

$$\prod_{k=1}^{n}\left(1 + \frac{t}{k}\frac{d}{dt}\right)\left[\frac{n}{t}\ F\left(\frac{n}{t}\right)\right] = \frac{n^{n+1}}{n!\ t^{n+1}} \int_0^\infty e^{-\frac{n}{t}x} x^n\ \ f(x)\, dx, \tag{14}$$

is generated, which in (10) gives the Tuan-Duc formula [6] as shown in (4).
Therefore, (3), (4) and (12) are equivalent among them because:

$$\frac{(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1}\left[\frac{d^n\ F}{d\ s^n}\right]_{s=\frac{n}{t}} = \prod_{k=1}^{n}\left(1 + \frac{t}{k}\frac{d}{dt}\right)\left[\frac{n}{t}\ F\left(\frac{n}{t}\right)\right] \tag{15}$$

$$= \frac{1}{(n-1)!\,t^{n+1}} \left(t^2 \frac{d}{d\,t} \right)^n F\left(\frac{n}{t}\right)$$

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