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ON SOME REFINEMENTS OF BERNSTEIN TYPE INEQUALITIES

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Abstract:
In this paper , we generalize some inequalities concerning to the Bernstein's inequality for polynomials .

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INTRODUCTION

If $P(z)$ is a polynomial of degree n, then concerning the estimate of the maximum of $|P(z)|$ on the unit circle $|z|=1$ and the estimate of the maximum of $|P(z)|$ on a large circle $|z|=R>1$, we have

$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)| \tag{1}$$

$$\max_{|z|=R>1}|P'(z)| \leq R^n \max_{|z|=1}|P(z)| \tag{2}$$

Inequality (1) is an immediate consequence of Bernstein's theorem on the derivative of a trigonometric polynomial (for reference see [4]). Inequality (2) is a simple deduction from maximum modulus principle (see [3,p.346] or [2,Vol. i, p.137]).
In both (1) and (2) equality holds only for $P(z)=\alpha z^n$, $|\alpha|\neq 0$ that is, if and only if $P(z)$ has all its zeros at the origin. Recently it was shown by Frappier, Rahman and Ruscheweyh [1,theorem 8] that, If $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=1}|P'(z)| \leq n \max_{1\leq k\leq 2n}\left|p(e^{\frac{ik\pi}{n}})\right| .$$

Clearly (3) represents a refinement of (1), since the maximum of $|P(z)|$ on may be larger than the maximum of $|P(z)|$ taken over the $(2n)^{th}$ roots of unity, as is shown by the example $P(z)=z^n+ia$, $a>0$

Now we have, If $P(z)$ is a polynomial of degree n, then for all real λ ¸ and R > 1,

$$\max_{|z|=1}|P(Rz)-P(z)| \leq \frac{R^n-1}{2}\left[M_\lambda + M_{\lambda+\pi}\right] \tag{4}$$

where

$$M_\lambda = \max_{1\le k\le n}\left|P(e^{\frac{i(2k\pi+\lambda)}{n}})\right|$$

and $M_{\lambda+\pi}$ is obtained by replacing λ by $\lambda+\pi$ in M_λ . The result is best possible and equality holds for $P(z)=z^n+re^{i\alpha}$, $-1\le r\le 1$.

Theorem A: If $P(z)$ is a polynomial of degree n , then for all real λ , and $R>1$,

$$\max_{|z|=1}|P(Rz)-P(z)| \le \frac{R^n-1}{2}[\ M_\lambda + M_{\lambda+\pi}\ ], \qquad (5)$$

where

$$M_\lambda = \max_{1\le k\le n}\left|P(e^{\frac{i(2k\pi+\lambda)}{n}})\right|$$

and $M_{\lambda+\pi}$ is obtained by replacing λ by $\lambda+\pi$ in M_λ .

Theorem B: If $P(z)$ is a polynomial of degree n, having all zeros in $|z|\ge 1$, then for all real λ and $R > 1$,

$$\max_{|z|=1}|P(Rz)-P(z)| \le \frac{R^n-1}{2}[\ M_\lambda{}^2 + M_{\lambda+\pi}{}^2\ ]^{1/2}. \qquad (6)$$

Theorem C: If $P(z)$ is a polynomial of degree n such that P(1)=0, then for $0\le\omega\le n$

$$|P\left(1-\frac{\omega}{n}\right)| \le [\left(1-\frac{\omega}{n}\right)]^n[\left|P\left(\frac{1}{r}\right)| - \frac{1}{2}\{M_0+M_{\lambda+\pi}\}\right] + \frac{1}{2}\{M_0+M_{\lambda+\pi}\}. \qquad (7)$$

MAIN RESULTS

Theorem 1 : If $P(z)$ is a polynomial of degree n, then for all real λ and $R>r\ge 1$

$$\max_{|z|=1}|P(rz)-P(rz)| \le \frac{R^n-r^n}{2}[\ M_\lambda + M_{\lambda+\pi}\ ] \qquad (8)$$

Remark 1: For $r=1$, we get (5).
On dividing both sides of (8) by $(R-r)$ and letting $R\to r$, we get
Corollary 1: If $P(z)$ is a polynomial of degree n , then for all real λ and $r\ge 1$

$$\max_{|z|=1}|P'(rz)| \le \frac{nr^{n-1}}{2}[\ M_\lambda + M_{\lambda+\pi}\ ]$$

Theorem 2: If $P(z)$ is a polynomial of degree n, having all zeros in $|z|\ge 1$, then for all real λ and $R>r\ge 1$,

$$\max_{|z|=1}\ |P(Rz)-P(rz)| \le \frac{R^n-r^n}{2}[\ M_\lambda{}^2 + M_{\lambda+\pi}{}^2\ ]^{1/2} \qquad (9)$$

Remark 2: For $r=1$, we get (6).
Now dividing on both sides by $(R-r)$ of (9) and letting $R\to r$, we obtain

Corollary 2: If $P(z)$ is a polynomial of degree n, having all zeros in $|z|\geq 1$, then for all real λ and $r\geq 1$,

$$\max_{|z|=1} |P'(rz)| \leq \frac{nr^{n-1}}{2}[\ M_{\lambda}^{\ 2} + M_{\lambda+\pi}^{\ \ 2}\ ]^{1/2}$$

Theorem 3: If $P(z)$ is a polynomial of degree n such that P(1)=0, then for $0\leq\omega\leq n$ $and\ r\geq 1$

$$|P\left(1-\frac{\omega}{n}\right)| \leq [\left(1-\frac{\omega}{n}\right)r]^n[|P\left(\frac{1}{r}\right)| - \frac{1}{2}\{M_0 + M_{\lambda+\pi}\}] + \frac{1}{2}\{M_0 + M_{\lambda+\pi}\}$$

Remark 3: For $r=1$, we get (7).

LEMMAS

To prove these results, we use the following lemmas:

Lemma 1: If $P(z)$ is a polynomial of degree n, then for all real λ,

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2}[\ M_{\lambda} + M_{\lambda+\pi}\ ],$$

where

$$M_{\lambda} = \max_{1\leq k\leq n}\left|P(e^{\frac{i(2k\pi+\lambda)}{n}})\right|$$

and $M_{\lambda+\pi}$ is obtained by replacing λ by $\lambda+\pi$ in M_{λ}.

Lemma 2: If $P(z)$ is a polynomial of degree n, having all zeros in $|z|\geq 1$, then for all real λ

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2}[\ M_{\lambda}^{\ 2} + M_{\lambda+\pi}^{\ \ 2}\ ]^{1/2}.$$

Type equation here.
The result is sharp and equality holds for

$$P(z) = z^n + e^{i\alpha}.$$

PROOF OF THEOREMS

Proof of Theorem 1: Applying (2) to the polynomial $P'(z)$, which is of degree $n-1$, we get

$$\left|p'(se^{i\vartheta})\right| \leq s^{n-1}\max_{|z|=1}\left|p'(z)\right|$$

Therefore by lemma 1, we have

$$\max_{|z|=1}|P'(sz)| \leq \frac{ns^{n-1}}{2}[\ M_{\lambda} + M_{\lambda+\pi}\ ]$$

Hence for each ϑ, $0\leq\vartheta<2\pi$ and $R>r\geq 1$, we have

$$\left|P(\mathrm{Re}^{i\vartheta}) - P(re^{i\vartheta})\right| = \left|\int_r^R e^{i\vartheta}P'(se^{i\vartheta})ds\right|$$

$$|P(Rz) - P(rz)| \leq \frac{n}{2}[\ M_{\lambda} + M_{\lambda+\pi}\ ]\int_r^R s^{n-1}ds$$

$$|P(Rz) - P(rz)| \leq \frac{R^n - r^n}{2}[\ M_{\lambda} + M_{\lambda+\pi}\ ],$$

where

$$M_{\lambda} = \max_{1\leq k\leq n}\left|P(e^{\frac{i(2k\pi+\lambda)}{n}})\right|$$

and $M_{\lambda+\pi}$ is obtained by replacing λ by $\lambda+\pi$ in M_{λ} .

Proof of Theorem 2: Applying (2) to the polynomial $P'(z)$ of degree $n-1$, we get

$$\left|p'(se^{i\vartheta})\right| \leq s^{n-1}\max_{|z|=1}\left|p'(z)\right|$$

Using lemma 2, we have

$$\max_{|z|=1}\ \left|P'(se^{i\vartheta})\right| \leq s^{n-1}(\tfrac{n}{2})[\ M_{\lambda}{}^2 + M_{\lambda+\pi}{}^2\ ]^{1/2}$$

Hence for each ϑ, $0 \leq \vartheta < 2\pi$ and $R > r \geq 1$, we have, for $R > r \geq 1$

$$\left|P(\mathrm{Re}^{i\vartheta}) - P(re^{i\vartheta})\right| = \left|\int_r^R e^{i\vartheta}P'(se^{i\vartheta})ds\right|.$$

This implies

$$\max_{|z|=1}\ |P(Rz) - P(rz)| \leq \frac{R^n - r^n}{2}[\ M_{\lambda}{}^2 + M_{\lambda+\pi}{}^2\ ]^{1/2},$$

where

$$M_{\lambda} = \max_{1\leq k\leq n}\left|P(e^{\frac{i(2k\pi+\lambda)}{n}})\right|$$

and $M_{\lambda+\pi}$ is obtained by replacing λ by $\lambda+\pi$ in M_{λ}.

Proof of Theorem 3: If $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$, then $|Q(z)| = |P(z)|$ for $|z| = 1$ andby hypothesis, we have $|Q(1)| = |\bar{P}(1)| = 0$. Applying theorem 1 to $Q(z)$ with λ=0, we get for $R >$r≥1

$$|Q(Rz) - Q(rz)| \leq \frac{R^n - r^n}{2}[\ M_0 + M_{\pi}\ ],$$

This implies for $R >$r ≥1

$$|\ \overline{P(\tfrac{1}{R})}\ | \leq \frac{1}{2R^n}\left(R^n - r^n\right)[\ M_0 + M_{\pi}\ ] + \frac{r^n}{R^n}|\overline{P(\tfrac{1}{r})}|.$$

If $0 < \omega \leq n$ then $\left(1 - \frac{\omega}{n}\right)^{-1} > 1$ and therefore and in particular, we have

$|\ P\left(1 - \frac{\omega}{n}\right)| \leq [\left(1 - \frac{\omega}{n}\right)r]^n[|P\left(\frac{1}{r}\right)| - \frac{1}{2}\{M_0 + M_{\lambda+\pi}\}] + \frac{1}{2}\{M_0 + M_{\lambda+\pi}\}$. Hence the result.

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