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SOME MATRIX TRANSFORMATIONS AND ALMOST CONVERGENCE

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ABSTRACT
The sequence space $bv(u,p)$ has been defined and the classes $(bv(u,p): l_\infty)$, $(bv(u,p): c)$ and $(bv(u,p): c_0)$ of infinite matrices have been characterized by Başar, Altay and Mursaleen (see, [2]). The main purposes of the present paper is to characterize the classes$(bv(u,p): f_\infty)$, $(bv(u,p): f)$ and$(bv(u,p): f_0)$, where f_∞, f and f_0 denotes the spaces of almost bounded sequences, almost convergent sequences and almost convergent null sequences, respectively, with real or complex terms.

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1 . INTRODUCTION, BACKGROUND AND PRELIMINARIES

A sequence space is defined to be a linear space with real or complex sequences. Throughout the paper $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the set of non-negative integers, the set of real numbers and the set of complex numbers, respectively. Let ℓ_∞, c and c_0 respectively be Banach spaces of bounded, convergent and null sequences $x = \{x_n\}_{n=0}^{\infty}$ normed by $\|x\| = sup_{n\geq 0}|x(n)|$; also, by cs we denote the sequence of all convergent series(see, [7]).

Let X and Y be two non-empty subsets of the space ω of real or complex sequences. Let $A = (a_{nk}), (n, k \in \mathbb{N})$, be an infinite matrix of real or complex numbers. We write $(Ax)_n = A_n(x) = \sum_k a_{nk} x_k$. Then $Ax = \{A_n(x)\}$ is called the A-transform of x, whenever $A_n(x) = \sum_k a_{nk} x_k$ converges for each $n \in \mathbb{N}$. We write $\lim_n Ax = \lim_n A_n(x)$. If $x \in X$ implies $Ax \in Y$, we say that A defines a (matrix) transformation from X into Y and we denote it by $A: X \to Y$. By $(X: Y)$, we mean the class of all matrices A such that $A: X \to Y$.

Let D denote the shift operator on ω, that is , $Dx = \{x(n)\}_{n=1}^{\infty}$, $D^2x = \{x(n)\}_{n=2}^{\infty}$ and so on. Obviously, D is a bounded linear operator on l_∞ onto itself. A Banach limit L is a non-negative linear functional on l_∞ such that L is invariant under the shift operator that is, $L(Sx) = L(x)$ and that $L(e) = 1$, where $e = \{1, 1, \ldots\}$ (see, [1]). A sequence space is said to be almost convergent (see, [3]) to the generalized limit α if all Banach limits of x are α. We denote the set of almost convergent sequences by f. It was proved by Lorentz (see, [3]) that

$$f = \{x \in l_\infty : \lim_m \tau_{mn}(x) = \alpha \,, uniformly\ in\ n\},$$

where, $\tau_{mn}(x) = \frac{1}{m+1}\sum_{j=0}^{m} x_{j+n}$, $\tau_{-1,n} = 0$ and $\alpha = f\text{-}\lim x$.

Nanda [6] has defined a new set of sequences f_∞ as follows:

$$f_\infty = \{x \in l_\infty : \lim_m |\tau_{mn}(x)| < \infty\}.$$

We call f_∞ the set of all almost bounded sequences.

We denote by X^β, the β-dual of a sequence space X and mean the set of all these sequences $x = (x_k)$ such that $xy = (x_k y_k) \in cs$ for all $y = (y_k) \in X$.

The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by several authors viz., ([2, 4, 5]).
The sequence space $bv(u,p)$ has been defined and the various classes $(bv(u,p)\colon l_\infty)$ $(bv(u,p)\colon c)$ and $(bv(u,p)\colon c_0)$ have been characterized (see, [2]). In the present paper, we characterize the classes $(bv(u,p)\colon f_\infty)$, $(bv(u,p)\colon f)$ and $(bv(u,p)\colon f_0)$, where $u = (u_k)$ is a sequence such that $u_k \neq 0$ for all $k \in \mathbb{N}$.

The space $bv(u,p)$ is defined (see, [2]) as

$$bv(u,p) = \{\, x = (x_k) \in \omega : \textstyle\sum_k |u_k \Delta x_k|^{p_k} < \infty \,\},$$

where, $\Delta x_k = x_k - \Delta x_{k-1}$.

2. MAIN RESULTS

Define the sequence $y = (y_k)$ which will be used as the A^u-transform of a sequence $x = (x_k)$, $i.e.$,

$$y_k = u_k \Delta x_k \; ; \; k \in \mathbb{N}. \tag{2.1}$$

For brevity in notation, we write

$$t_{mn}(x) = \frac{1}{m+1} \sum_{j=0}^{m} A_{n+j}(x) = \sum_k a(n,k,m) x_k,$$

where , $$a(n,k,m) = \frac{1}{m+1} \sum_{j=0}^{m} a_{n+j,k} \quad ; (n,k,m \in \mathbb{N})$$

Also, $$\bar{a}(n,k,m) = \left[\frac{a(n,k,m)}{u_k}\right] ; (n,k,m \in \mathbb{N}).$$

Now, we give the following lemmas which will be needed in proving the main Theorems.

Lemma 2.1 [2] : Define the sets $D_1(p)$and $D_2(p)$ as follows:

$$D_1(p) = \left\{ a = (a_k) \in \omega : \sup_n \sum_k \left| \sum_{j=k}^{n} \frac{a_j}{u_k} \right|^{p_k} < \infty \right\},$$

$$D_2(p) = \bigcup_{B>1} \left\{ a = (a_k) \in \omega : \sup_n \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \frac{a_j}{u_k} B^{-1} \right|^{p_k'} < \infty \right\}.$$

Then , $[bv(u,p)]^\beta = D_1(p) \cap cs$; $(0 < p_k \leq 1)$
and $[bv(u,p)]^\beta = D_2(p) \cap cs$; $(1 < p_k < \infty)$.

Lemma 2.2 [6]: $f \subset f_\infty$.

We consider only the case $1 < p_k \leq M < \infty$ and the case $0 < p_k \leq 1$ may be proved in a similar fashion.

Theorem 2.3: (a) Let $1 < p_k \leq M < \infty$ for every $k \in \mathbb{N}$. Then $A \in (bv(u,p)\colon f_\infty)$ if and only if

$$\sup_{n,m} \sum_k |\bar{a}(n,k,m) B^{-1}|^{p_k'} < \infty \tag{2.2}$$

and $$\{a_{nk}\} \in D_2(p) \cap cs. \tag{2.3}$$

(b) Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A \in (bv(u,p): f_\infty)$ if and only if

$$\sup_{n,m} \sum_k |\bar{a}(n,k,m)|^{p_k} < \infty \tag{2.4}$$

and $$\{a_{nk}\} \in D_1(p) \cap cs. \tag{2.5}$$

Proof : Sufficiency: Suppose the conditions holds and $x \in bv(u,p)$.Using the inequality which holds for any $C > 0$ and any two complex numbers a, b

$$|ab| \leq C\{|aC^{-1}|^q + b^p\},$$

where, $p > 1$ and $p^{-1} + q^{-1} = 1$ (see, [3]), we have

$$|t_{mn}(Ax)| = |\textstyle\sum_k a(n,k,m)x_k| = |\sum_k \bar{a}(n,k,m)y_k|$$
$$\leq \textstyle\sum_k B\left[|\bar{a}(n,k,m)B^{-1}|^{p_k'} + |y_k|^{p_k}\right]$$

Now, taking sup over m, n on both sides to the above inequality, we get $Ax \in f_\infty$ for every $x \in bv(u,p)$, $i.e.$, $A \in (bv(u,p): f_\infty)$.

Necessity: Suppose that $A \in (bv(u,p): f_\infty)$. Then Ax exists for every $x \in bv(u,p)$, and this implies that $\{a_{n,k}\}_{k\in\mathbb{N}} \in [bv(u,p)]^\beta$ for every $n \in \mathbb{N}$, the necessity of (2.3) is immediate.

Now, $\sum_k a(n,k,m)x_k$ exists for each m, n and $x \in bv\ (u,p)$, the sequences $\{a(n,k,m)\}_{k\in\mathbb{N}}$ define the continuous linear functionals $\varphi_{mn}(x)$ on $bv(u,p)$ by $\varphi_{mn}(x) = \sum_k a(n,k,m)x_k$; $n,k,m \in \mathbb{N}$. Since $bv(u,p)$is complete and $\sup_{m,n}|\sum_k \bar{a}(n,k,m)x_k| < \infty$, so by uniform bounded principle , there exists $M > 0$ such that

$$sup_{m,n}|\varphi_{mn}(x)| = sup_{m,n}|\textstyle\sum_k a(n,k,m)x_k|$$
$$= sup_{m,n}|\textstyle\sum_k \bar{a}(n,k,m)x_k| \leq M < \infty.$$

This implies that $sup_{m,n}\sum_k |\bar{a}(n,k,m)x_k|^{p_k'} < \infty$, which shows the necessity of the condition (2.2) and the proof of (i) is complete.

Theorem 2.4 : (a) Let $1 < p_k \leq M < \infty$ for every $k \in \mathbb{N}$. Then $A \in (bv(u,p): f_\infty)$ if and only if (i) the condition (2.2)-(2.5) of Theorem 2.3 holds

(ii) there is a sequence (β_k) of scalars such that

$$lim_m \bar{a}(n,k,m) = \beta_k, \text{ uniformly in } n. \tag{2.6}$$

Proof: Sufficiency: Suppose that the conditions (2.2)-(2.6) hold and $x \in bv(u,p)$. Then Ax exists and we have by (2.6) that $|\bar{a}(n,k,m)B^{-1}|^{p_k'} \to |\beta_k B^{-1}|^{p_k'}$ as $m \to \infty$ uniformly in n for each $k \in \mathbb{N}$, which leads us with (2.2) that

$$\textstyle\sum_{j=0}^k |\beta_j B^{-1}|^{p_k'} = \sum_{j=0}^k |\bar{a}(n,j,m)B^{-1}|^{p_k'}$$
$$\leq sup_{m,n}\textstyle\sum_j |\bar{a}(n,j,m)B^{-1}|^{p_k'} < \infty,$$

holding for every $k \in \mathbb{N}$. Consequently reasoning as in the proof of the sufficiency of Theorem 2.3, the series $\sum_k a(n,k,m)x_k$ and $\sum_k \beta_k x_k$ converges for every n, m and for every $x \in bv(u,p)$. Now, for given $\varepsilon > 0$ and $x \in bv(u,p)$, choose a fixed $k_0 \in \mathbb{N}$ such that

$\left[\sum_{k=k_0+1}^{\infty}|x_k|^{p_k}\right]^{\frac{1}{H}} < \varepsilon$, where $H = sup_k p_k$. Then, there is some $m_0 \in \mathbb{N}$, by condition (ii) such that $\left|\sum_{k=1}^{k_0}[a(n,k,m) - \beta_k]\right| < \varepsilon$, for every $m \geq m_0$ and uniformly in n.

Now, since $\sum_k a(n,k,m)x_k$ and $\sum_k \beta_k x_k$ converges (absolutely) uniformly in n, m and for $x \in bv(u,p)$, we have that $\sum_{k_0+1}^{\infty}[\,a(n,k,m) - \beta_k]\,x_k < \frac{\varepsilon}{2}$, converges uniformly in n, m and $x \in bv(u,p)$. Hence by conditions (i) and (ii) we have $\sum_{k_0+1}^{\infty}[\,a(n,k,m) - \beta_k] < \frac{\varepsilon}{2}$ for all $(m \geq m_0)$, uniformly in n. Therefore, $\left|\sum_{k_0+1}^{\infty}[\,a(n,k,m) - \beta_k]\right| \to 0\ (m \to \infty)$ uniformly in $i.e.$,

$$lim_m \sum_k a(n,k,m)x_k = \sum_k \beta_k x_k \text{ uniformly in } n. \qquad (2.7)$$

Hence, $Ax \in f$, which proves sufficiency.

Necessity: Suppose that $A \in (\,bv(u,p){:}\,f\,)$. Then, since $f \subset f_\infty$(by Lemma 2.1), the necessities of condition (i) is immediately obtained from Theorem 2.1 . To prove the necessity of (ii) $i.e.$, (2.6), consider the sequence $e_k = (\,0,0,\ldots,1^{kth-place},0,0,\ldots\,) \in bv(u,p)$, condition (ii) follows immediately by (2.7) and the proof is complete.

Collary 2.5: $A \in (\,bv(u,p){:}\,f_0)$ if and only if condition (i) and (ii) of above Theorem holds along with $\beta_k = 0$ for each $k \in \mathbb{N}$.

Proof: The proof follows from theorem 2.4 by taking $\beta_k = 0$ for each $k \in \mathbb{N}$.

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