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A COMMON FIXED POINT OF WEAKLY COMPATIBLE MAPPINGS IN DISLOCATED METRIC SPACE

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ABSTRACT
The purpose of this paper is to establish a common fixed point theorem for pairs of weakly compatible maps in dislocated metric space which generalizes and improves similar fixed point results.

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1 INTRODUCTION
In 2000, P. Hitzler and A. K. Seda [3] introduced the notion of dislocated metric space and generalized the famous Banach contraction principle in this space. Dislocated metric space plays very important role in topology, logical programming and in electronics engineering. C. T. Aage and J. N. Salunke [1], A. Isufati [5] established some important fixed point theorems for single and pair of mappings. K. Jha et.al. [8] and K. Jha and D. Panthi [6,7] have recently established common fixed point theorems for two pairs of weakly compatible mappings in dislocated metric space. Our result generalizes and improves some similar results of fixed points in this space.

2 PRELIMINARIES AND DEFINITIONS
Now, we start with the following definitions, lemmas and theorems.

Definition 2.1 [13]: Let X be a non empty set and let $d : X \times X \rightarrow [0, \infty)$ be a function satisfying the following conditions:

(i) $d(x, y) = d(y, x)$

(ii) $d(x, y) = d(y, x) = 0$ implies $x = y$.

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in$ X.

Then d is called dislocated metric (or simply d-metric) on X.

Definition 2.2 [3]: A sequence $\{x_n\}$ in a d-metric space (X, d) is called a Cauchy sequence if for given $\epsilon > 0$, there corresponds $n_0 \in$ N such that for all m, n $\geq n_0$, we have $d(x_m, x_n) < \epsilon$.

Definition 2.3 [3]: A sequence in d-metric space converges with respect to d (or in d) if there exists $x \in X$ such that $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.

In this case, x is called limit of $\{x_n\}$ (in d) and we write $x_n \rightarrow$ x.

Definition 2.4 [3]: A d-metric space (X,d) is called complete if every Cauchy sequence in it is convergent with respect to d.

Definition 2.5 [3]: Let (X,d) be a d-metric space. A map $T:X\to X$ is called contraction if there exists a number λ with $0 \le \lambda < 1$ such that $d(Tx,Ty) \le \lambda\ d(x,y)$.

We state the following lemmas without proof.

Lemma 2.1: Let (X,d) be a d-metric space. If $T:X\to X$ is a contraction function, then $\{T^n(x_0)\}$ is a Cauchy sequence for each $x_0 \in$ X.

Lemma 2.2 [3]: Limits in a d-metric space are unique.

Definition 2.6 [9]: Let A and S be mappings from a metric space (X,d) into itself. Then, A and S are said to be weakly compatible if they commute at their coincident point; that is, $Ax = Sx$ for some $x\in X$ implies $ASx = SAx$.

Theorem 2.1 [3]: Let (X,d) be a complete d-metric space and let $T:X\to X$ be a contraction mapping, then T has a unique fixed point.

3 MAIN RESULTS

Theorem 3.1: Let (X,d) be a complete d-metric space. Let A, B, S, T: X$\to X$ be continuous mappings satisfying,

(i) $T(X)\subset A(X),\quad S(X)\subset B(X)$

(ii) The pairs (S, A) and (T, B) are weakly compatible and

(iii) $d(Sx,Ty)\le \alpha[d(Ax,Ty)+d(By,Sx)]+\beta[d(By,Ty)+d(Ax,Sx)]+\gamma d(Ax,By)$

for all x, y $\in X$ where $\alpha,\beta,\gamma \ge 0,\quad 0\le \alpha+\beta+\gamma<\frac{1}{4}$.

Then A, B, S, and T have a unique common fixed point.

Proof:

Using condition (i), we define sequences $\{x_n\}$ and $\{y_n\}$ in X by the rule,

$$y_{2n} = Bx_{2n+1} = Sx_{2n},\quad and y_{2n+1} = Ax_{2n+2} = Tx_{2n+1}, n = 0,1,2...$$

If $y_{2n} = y_{2n+1}$ for some n, then $Bx_{2n+1} = Tx_{2n+1}$.Therefore x_{2n+1} is a coincidence point of B and T. Also, if $y_{2n+1} = y_{2n+2}$ for some n, then $Ax_{2n+2} = Sx_{2n+2}$. Hence x_{2n+2} is a coincidence point of S and A. Assume that $y_{2n} \ne y_{2n+1}$ for all n. Then, we have

$$\begin{aligned}
d(y_{2n},y_{2n+1}) &= d(Sx_{2n},Tx_{2n+1})\\
&\le \alpha[d(Ax_{2n},Tx_{2n+1})+d(Bx_{2n+1},Sx_{2n})]+\beta[d(Bx_{2n+1},Tx_{2n+1})+d(Ax_{2n},Sx_{2n})]\\
&\qquad +\gamma d(Ax_{2n},Bx_{2n+1})\\
&\le \alpha[d(y_{2n-1},y_{2n+1})]+d(y_{2n},y_{2n})]+\beta[d(y_{2n},y_{2n+1})+d(y_{2n-1},y_{2n})]\\
&\qquad +\gamma d(y_{2n-1},y_{2n})\\
&\le \alpha[d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})+d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})]\\
&\qquad +\beta[d(y_{2n},y_{2n+1})+d(y_{2n-1},y_{2n})]+\gamma d(y_{2n-1},y_{2n})\\
&= (2\alpha+\beta+\gamma)d(y_{2n-1},y_{2n})+(2\alpha+\beta)d(y_{2n},y_{2n+1})
\end{aligned}$$

Therefore,

$$d(y_{2n}, y_{2n+1}) \le \frac{2\alpha+\beta+\gamma}{1-2\alpha-\beta} d(y_{2n-1}, y_{2n})$$
$$= hd(y_{2n-1}, y_{2n})$$

where,

$$h = \frac{2\alpha+\beta+\gamma}{1-2\alpha-\beta} < 1$$

This shows that,

$$d(y_n, y_{n+1}) \le hd(y_{n-1}, y_n) \le ... \le h^n d(y_0, y_1)$$

Thus, for every integer $q > 0$, we have

$$d(y_n, y_{n+q}) \le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + ... + d(y_{n+q-1}, y_{n+q})$$
$$\le (1+h+h^2+...+h^{q-1}) d(y_n, y_{n+1})$$
$$\le \frac{h^n}{1-h} d(y_0, y_1)$$

Since, $0 < h < 1, h^n \to 0 \quad as \quad n \to \infty.$

So, we get $d(y_n, y_{n+q}) \to 0.$ This implies that $\{y_n\}$ is a Cauchy sequence in a complete dislocated metric space. So, there exists a point $z \in X$ such that $\{y_n\} \to z.$
Therefore, the subsequences,

$$\{Sx_{2n}\} \to z, \{Bx_{2n+1}\} \to z, \{Tx_{2n+1}\} \to z \quad and \quad \{Ax_{2n+2}\} \to z.$$

Since $T(X) \subset A(X),$ there exists a point $u \in X$ such that $z = Au.$ So,

$$d(Su, z) = d(Su, Tx_{2n+1})$$
$$\le \alpha[d(Au, Tx_{2n+1}) + d(Bx_{2n+1}, Su)] + \beta[d(Bx_{2n+1}, Tx_{2n+1}) + d(Au, Su)] + \gamma d(Au, Bx_{2n+1})$$
$$= \alpha[d(z, Tx_{2n+1}) + d(Bx_{2n+1}, Su)] + \beta[d(z, Su)] + \gamma d(z, Bx_{2n+1})$$

Now, taking limit as n $\to$ ∞, we get, $d(Su, z) < \beta d(z, Su)$
which is a contradiction.

So, we have $Su = Au = z.$
Again, since $S(X) \subset B(X)$, there exists a point $v \in X$ such that $z = Bv.$
We claim that $z = Tv.$ If $z \neq$ Tv, then

$$d(z, Tv) = d(Su, Tv)$$
$$\le \alpha[d(Au, Tv) + d(Bv, Su)] + \beta[d(Bv, Tv) + d(Au, Su)] + \gamma d(Au, Bv)$$
$$= \alpha[d(z, Tv) + d(z, z)] + \beta[d(z, Tv) + d(z, z)] + \gamma d(z, z)$$
$$= (3\alpha + 3\beta + 2\gamma) d(z, Tv)$$

a contradiction. So, we get $z = Tv.$
Hence, we have $Su = Au = Tv = Bv = z.$

Since the pair (S, A) are weakly compatible so by definition $SAu = ASu$ implies $Sz = Az.$
Now, we show that z is the fixed point of S . If Sz $\neq z$, then

$$\begin{aligned} d(Sz,z) &= d(Sz,Tv) \\ &\le \alpha[d(Az,Tv)+d(Bv,Sz)]+\beta[d(Bv,Tv)+d(Az,Sz)]+\gamma d(Az,Bv) \\ &= \alpha[d(Sz,z)+d(z,Sz)]+\beta[d(z,z)+d(Sz,Sz)]+\gamma d(Sz,z) \\ &\le (2\alpha+4\beta+\gamma)d(Sz,z) \end{aligned}$$

which is a contradiction. So, we have $Sz = z.$
This implies that $Az = Sz = z.$

Again, the pair (T, B) are weakly compatible, so by definition $TBv = BTv$ implies $Tz = Bz.$
Now, we show that z is the fixed point of T. If $Tz \neq z$, then

$$\begin{aligned} d(z,Tz) &= d(Sz,Tz) \\ &\le \alpha[d(Az,Tz)+d(Bz,Sz)]+\beta[d(Bz,Tz)+d(Az,Sz)]+\gamma d(Az,Sz) \\ &= \alpha[d(z,Tz)+d(Tz,z)]+\beta[d(Tz,Tz)+d(z,z)]+\gamma d(z,Tz) \\ &\le (2\alpha+4\beta+2\gamma)d(z,Tz) \end{aligned}$$

which is a contradiction. This implies that $z = Tz.$
Hence, we have $Az = Bz = Sz = Tz = z.$

This shows that z is the common fixed point of the self mappings A, B, S and T.

Uniqueness:
Let $u \neq v$ be two common fixed points of the mappings A, B, S and T . Then we have,

$$\begin{aligned} d(u,v) &= d(Su,Tv) \\ &\le \alpha[d(Au,Tv)+d(Bv,Su)]+\beta[d(Bv,Tv)+d(Au,Su)]+\gamma d(Au,Bv) \\ &= \alpha[d(u,v)+d(v,u)]+\beta[d(v,v)+d(u,u)]+\gamma d(u,v) \\ &= (2\alpha+4\beta+\gamma)d(u,v) . \end{aligned}$$

a contradiction. This shows that $d(u,v) = 0$
Since (X,d) is a dislocated metric space, so we have $u = v.$ This establishes the theorem.

From above theorem we can obtain the following corollaries.

Corollary 3.1: Let (X,d) be a complete d-metric space. Let S, T: X $\rightarrow X$ be continuous mappings satisfying,
$d(Sx,Ty) \le \alpha[d(x,Ty)+d(y,Sx)]+\beta[d(y,Ty)+d(x,Sx)]+\gamma d(x,y)$ for all x, y $\in X$, where $\alpha,\beta,\gamma \ge 0, \quad 0 \le \alpha+\beta+\gamma < \frac{1}{4}.$ Then S and T have a unique common fixed point.

Proof: If we take A = B = I an identity mapping in theorem 3.1, and follow the similar proof as that in the theorem, we can establish this corollary.

If we take $S = T$ then the above corollary is reduced to,

Corollary 3.2: Let (X,d) be a complete d-metric space. Let $T: X \rightarrow X$ be a continuous mapping satisfying,
$d(Tx,Ty) \le \alpha[d(x,Ty)+d(y,Tx)]+\beta[d(y,Ty)+d(x,Tx)]+\gamma d(x,y)$ for all x, y $\in X$,where $\alpha,\beta,\gamma \ge 0, \quad 0 \le \alpha+\beta+\gamma < \frac{1}{4}.$ Then T has a unique fixed point.

If we put $\beta = 0$ and rearrange the constants, then we can obtain the theorem 3.2 established by A. Isufati [5].

Corollary 3.3: Let (X, d) be a complete d-metric space. Let S, T :X $\to X$ be continuous mappings satisfying,
$d(Sx,Ty) \le \alpha[d(Tx,Ty) + d(Sy,Sx)] + \beta[d(Sy,Ty) + d(Tx,Sx)] + \gamma d(Tx,Sy)$ for all x, y $\in X$,where $\alpha, \beta, \gamma \ge 0, \quad 0 \le \alpha + \beta + \gamma < \frac{1}{4}$. Then S and T have a unique common fixed point.

Proof: If we take $A = T$ and $B = S$ in above Theorem 3.1 and apply the similar proof, we can establish this corollary 3.3.

Remarks: Our result generalizes the results of A. Isufati [5] and improves the results of C. T. Aage and J. N. Salunke [1, 2], R. Shrivastava et.al. [11], K. Jha and D. Panthi [6, 7], K. P. R. Rao and P. Rangaswamy [10], K. Jha et. al. [8] and similar other results of fixed point in the literature.

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