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A COMMON FIXED POINT THEOREM FOR SIX EXPANSIVE MAPPINGS IN G – METRIC SPACES

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ABSTRACT:
In this paper we obtain a unique common fixed point theorem for six expansive mappings in G –metric spaces.

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Key words : Expansive mappings, G –metric space, Weakly compatible mappings.

1. INTRODUCTION

Dhage [2, 3, 4, 5]. et al. introduced the concept of D –metric spaces as generalization of ordinary metric functions and went on to present several fixed point results for single and multivalued mappings. Mustafa and Sims [6] and Naidu et al. [10, 11, 12] demonstrated that most of the claims concerning the fundamental topological structure of D – metric space are incorrect, alternatively, Mustafa and Sims introduced in [6] more appropriate notion of generalized metric space which called G – metric spaces, and obtained some topological properties. Later Zead Mustafa, Hamed Obiedat and Fadi Awawdeh[7], Mustafa, Shatanawi and Bataineh [8], Mustafa and Sims [9] Shatanawi [13] and Renu Chugh, Tamanna Kadian, Anju Rani and B.E. Rhoades [1] et al. obtained some fixed point theorems for a single map in G- metric spaces. In this paper, we obtain a unique common fixed point theorem for six weakly compatible expansive mappings in G – metric spaces . First, we present some known definitions and propositions in G – metric spaces .

DEFINITION 1.1 [6] : Let X be a nonempty set and let $G: X \times X \times X \to R^+$ be a function satisfying the following properties :

(G_1) : $G(x, y, z) = 0$ if $x = y = z$,

(G_2) : $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G_3) : $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G_4) : $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, symmetry in all three variables,

(G_5) : $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then the function G is called a generalized metric or a G – metric on X and the pair (X, G) is called a G- metric space.

DEFINITION 1.2 [6] : Let (X, G) be a G- metric space and $\{x_n\}$ be a sequence in X. A point $x \in X$ is said to be limit of $\{x_n\}$ iff $\lim_{n, m\to\infty} G(x, x_n, x_m) = 0$. In this case, the sequence $\{x_n\}$ is said to be G – convergent to x.

DEFINITION 1.3 [6] : Let (X, G) be a G- metric space and $\{x_n\}$ be a sequence in X. $\{x_n\}$ is called G- Cauchy iff $\lim_{n, m\to\infty} G(x_l, x_n, x_m) = 0$. (X, G) is called G –complete if every G–Cauchy sequence in (X, G) is G-convergent in (X, G).

PROPOSITION 1.4 [6] : In a G- metric space,(X, G), the following are equivalent.

(1) The sequence $\{x_n\}$ is G- Cauchy.

(2) For every $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.

PROPOSITION 1.5 [6] : Let (X, G) be a G- metric space. Then the function G (x, y, z) is jointly continuous in all three of its variables.

PROPOSITION 1.6 [6] : Let (X, G) be a G- metric space. Then for any $x, y, z, a \in X$, it follows that

(i) if $G(x, y, z) = 0$ then $x = y = z$,

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,

(iii) $G(x, y, y) \leq 2G(x, x, y)$,

(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,

(v) $G(x, y, z) \leq \frac{2}{3}[G(x, a, a) + G(y, a, a) + G(z, a, a)]$.

PROPOSITION 1.7 [6] : Let (X, G) be a G- metric space. Then for a sequence $\{x_n\} \subseteq X$ and a point $x \in X$, the following are equivalent

(i) $\{x_n\}$ is G- convergent to x,

(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,

(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$,

(iv) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

2. RESULTS

THEOREM 2.1: Let (X, G) be a complete G- metric space and

S, T, R, f, g, h : $X \to X$ be mappings such that

(2.1.1) $$G(Sx, Ty, Rz) \geq q \max\left\{\begin{matrix} G(fx, gy, hz), G(fx, Sx, Rz), \\ G(gy, Ty, Sx), G(hz, Rz, Ty) \end{matrix}\right\}$$

for all $x,y,z \in X$ and $q > 1$,

(2.1.2) $h(X) \subseteq S(X)$, $f(X) \subseteq T(X)$, $g(X) \subseteq R(X)$,

(2.1.3) one of f(X) , g(X) and h(X) is a G- complete subspace of X ,

(2.1.4) the pairs (f, S), (g, T) and (h, R) are weakly compatible.

Then (a) one of the pairs (f, S), (g, T) and (h, R) has a coincidence point in X or

(b) S, T, R, f, g and h have a unique common fixed point in X.

PROOF : Let $x_0 \in X$.

From (2.1.2), there exist $x_1, x_2, x_3 \in X$ such that $hx_0 = Sx_1 = y_1$, say,

$fx_1 = Tx_2 = y_2$, say and $gx_2 = Rx_3 = y_3$, say .

By induction, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$hx_{3n} = Sx_{3n+1} = y_{3n+1}$, $fx_{3n+1} = Tx_{3n+2} = y_{3n+2}$, $gx_{3n+2} = Rx_{3n+3} = y_{3n+3}$, n = 0, 1, 2,…

If $y_{3n+1} = y_{3n+2}$ then $f x = S x$, where $x = x_{3n+1}$.

If $y_{3n+2} = y_{3n+3}$ then $g x = T x$, where $x = x_{3n+2}$.

If $y_{3n} = y_{3n+1}$ then $h x = R x$, where $x = x_{3n}$.

Assume that $y_n \neq y_{n+1}$ for all n .

Denote $d_n = G(y_n, y_{n+1}, y_{n+2})$.

$d_{3n-1} = G(y_{3n-1}, y_{3n}, y_{3n+1})$

$\quad = G(Tx_{3n-1}, Rx_{3n}, Sx_{3n+1})$

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$$\geq q \max \begin{Bmatrix} G(y_{3n+2}, y_{3n}, y_{3n+1}), G(y_{3n+2}, y_{3n+1}, y_{3n}) \\ G(y_{3n}, y_{3n-1}, y_{3n+1}), G(y_{3n+1}, y_{3n}, y_{3n-1}) \end{Bmatrix}$$

$$= q \max \{d_{3n}, d_{3n}, d_{3n-1}, d_{3n-1}\} .$$

Thus we have $d_{3n-1} \geq q\, d_{3n}$ so that $d_{3n} \leq k\, d_{3n-1}$, where $k = \frac{1}{q} < 1$.

$$d_{3n} = G(y_{3n}, y_{3n+1}, y_{3n+2})$$

$$= G(Rx_{3n}, Sx_{3n+1}, Tx_{3n+2})$$

$$\geq q \max \begin{Bmatrix} G(y_{3n+2}, y_{3n+3}, y_{3n+1}), G(y_{3n+2}, y_{3n+1}, y_{3n}) \\ G(y_{3n+3}, y_{3n+2}, y_{3n+1}), G(y_{3n+1}, y_{3n}, y_{3n+2}) \end{Bmatrix}$$

$$= q \max \{d_{3n+1}, d_{3n}, d_{3n+1}, d_{3n}\} .$$

Thus we have $d_{3n} \geq q\, d_{3n+1}$ so that $d_{3n+1} \leq k\, d_{3n}$.

$$d_{3n+1} = G(y_{3n+1}, y_{3n+2}, y_{3n+3})$$

$$= G(Sx_{3n+1}, Tx_{3n+2}, Rx_{3n+3})$$

$$\geq q \max \begin{Bmatrix} G(y_{3n+2}, y_{3n+3}, y_{3n+4}), G(y_{3n+2}, y_{3n+1}, y_{3n+3}) \\ G(y_{3n+3}, y_{3n+2}, y_{3n+1}), G(y_{3n+4}, y_{3n+3}, y_{3n+2}) \end{Bmatrix}$$

$$= q \max \{d_{3n+2}, d_{3n+1}, d_{3n+1}, d_{3n+2}\} .$$

Thus we have $d_{3n+1} \geq q\, d_{3n+2}$ so that $d_{3n+2} \leq k\, d_{3n+1}$.

Hence
$$\begin{aligned} G(y_n, y_{n+1}, y_{n+2}) &\leq k\, G(y_{n-1}, y_n, y_{n+1}) \\ &\leq k^2 G(y_{n-2}, y_{n-1}, y_n) \\ &\;\;\vdots \\ &\;\;\vdots \\ &\leq k^n G(y_0, y_1, y_2). \end{aligned}$$

From (G_3), we have

$$G(y_n, y_n, y_{n+1}) \leq G(y_n, y_{n+1}, y_{n+2}) \leq k^n G(y_0, y_1, y_2).$$

From (G_5) for $m > n$ we have

$$\begin{aligned} G(y_n, y_n, y_m) &\leq G(y_n, y_n, y_{n+1}) + G(y_{n+1}, y_{n+1}, y_{n+2}) + \ldots + G(y_{m-1}, y_{m-1}, y_m) \\ &\leq (k^n + k^{n+1} + \ldots + k^{m-1})\, G(y_0, y_1, y_2) \end{aligned}$$

$$\leq \frac{k^n}{1-\alpha} G(y_0, y_1, y_2)$$

$$\to 0 \text{ as } n \to \infty, m \to \infty.$$

Hence $\{y_n\}$ is G- Cauchy.

Suppose f(X) is a G-complete subspace of X. Then there exist p, t $\in$ X such that $y_{3n+1} \to p = f\,t$.

Since $\{y_n\}$ is G- Cauchy, it follows that $y_{3n} \to p$ and $y_{3n+2} \to p$.

$G(St, y_{3n+2}, y_{3n+3}) = G(St, Tx_{3n+2}, Rx_{3n+3})$

$$\geq q \max \begin{Bmatrix} G(p, y_{3n+3}, y_{3n+4}), G(p, St, y_{3n+3}) \\ G(y_{3n+3}, y_{3n+2}, St), G(y_{3n+4}, y_{3n+3}, y_{3n+2}) \end{Bmatrix}.$$

Letting $n \to \infty$, we get

$G(St, p, p) \geq G(p, St, p)$.

Hence St = p. Thus f t = St = p.

Since (f, S) is a weakly compatible pair, we have f p = Sp.

$G(Sp, y_{3n+2}, y_{3n+3}) = G(Sp, Tx_{3n+2}, Rx_{3n+3})$

$$\geq q \max \begin{Bmatrix} G(Sp, y_{3n+3}, y_{3n+4}), G(Sp, Sp, y_{3n+3}) \\ G(y_{3n+3}, y_{3n+2}, Sp), G(y_{3n+4}, y_{3n+3}, y_{3n+2}) \end{Bmatrix}.$$

Letting $n \to \infty$, we get

$G(Sp, p, p) \geq q \max \{ G(Sp, p, p), G(Sp, Sp, p), G(p, p, Sp), 0\}$

$$\geq q \max\left\{G(Sp, p, p), \frac{1}{2} G(Sp, p, p), 0\right\}, \text{ since } G(p, p, Sp) \leq 2G(Sp, Sp, p)$$

$$= q\, G(Sp, p, p).$$

Hence Sp = p. Thus f p = Sp = p. ……(1)

Since p = Sp $\in$ T(X), there exists v $\in$ X such that p = Tv.

$G(Sp, Tv, y_{3n+3}) = G(Sp, Tv, Rx_{3n+3})$

$$\geq q \max \{G(p, gv, y_{3n+4}), G(p, p, y_{3n+3}), G(gv, p, p), G(y_{3n+4}, y_{3n+3}, p)\}.$$

Letting $n \to \infty$ we get, $0 \geq q \max \{G(p, gv, p), 0, G(gv, p, p), 0)\}$.

Hence G(p, gv, p) = 0 so that gv = p. Thus gv = Tv = p.

Since (g, T) is a weakly compatible pair, we have g p = Tp.

$G(p, Tp, y_{3n+3}) = G(Sp, Tv, Rx_{3n+3})$

$$\geq q \max \left\{ \begin{matrix} G(p, Tp, y_{3n+4}), G(p, p, y_{3n+3}), \\ G(Tp, Tp, p), G(y_{3n+4}, y_{3n+3}, Tp) \end{matrix} \right\}.$$

Letting $n \to \infty$ we get

$G(p, Tp, p) \geq q \max \{G(p, Tp, p), 0, G(Tp, Tp, p), G(p, p, Tp)\}$

$$\geq q \max \left\{ G(p, Tp, p), \frac{1}{2} G(p, p, Tp) \right\}, \text{ since } G(p, p, Tp) \leq 2\, G(Tp, Tp, p)$$

$$= q\, G(p, p, Tp).$$

Hence $Tp = p$. Thus $g\,p = Tp = p$. ……(2)

Since $p = gp \in R(X)$, there exists $w \in X$ such that $p = hw$.

$G(p, p, Rw) = G(Sp, Tp, Rw)$

$$\geq q \max \{G(p, p, p), G(p, p, Rw), G(p, p, p), G(p, Rw, p)\}$$

$$= q\, G(p, p, Rw).$$

Hence $Rw = p$. Thus $hw = Rw = p$.

Since (h, R) is a weakly compatible pair, we have $Rp = hp$.

$G(p, p, Rp) = G(Sp, Tp, Rp)$

$$\geq q \max \{G(p, p, Rp), G(p, p, Rp), G(p, p, p), G(Rp, Rp, p)\}$$

$$\geq q \max \left\{ G(p, p, Rp), \frac{1}{2} G(p, p, Rp) \right\}, \text{ since } G(p, p, Rp) \leq 2\, G(Rp, Rp, p)$$

$$= q\, G(p, p, Rp).$$

Hence $Rp = p$. Thus $hp = Rp = p$. ……(3)

From (1), (2) and (3) it follows that p is a common fixed point of S, T, R, f, g and h.

Suppose p' is another common fixed point of S, T, R, f, g and h.

$G(p, p, p') = G(Sp, Tp, Rp')$

$$\geq q \max \{G(p, p, p'), G(p, p, p'), G(p, p, p), G(p', p', p)\}$$

$$\geq q \max \left\{ G(p, p, p'), \frac{1}{2} G(p, p, p') \right\}, \text{ since } G(p, p, p') \leq 2\, G(p', p', p)$$

$$= q\, G(p, p, p').$$

Hence $p' = p$.

Thus p is a unique common fixed point of S, T, R, f, g and h.

Similarly, the theorem holds if g(X) or h(X) is a G- complete subspace of X.

Finally, we prove the following in the similar lines.

THEOREM 2.2: Let (X, G) be a complete G- metric space and

S, T, R, f, g, h : $X \to X$ be mappings such that

(2.2.1) $$G(Sx, Ty, Rz) \geq q \min \left\{ \begin{array}{l} G(fx, gy, hz), G(fx, Sx, Rz), \\ G(gy, Ty, Sx), G(hz, Rz, Ty) \end{array} \right\}$$

or

$$G(Sx, Ty, Rz) \geq q\ G(fx,gy,hz)$$

for all $x,y,z \in X$ and $q > 1$,

(2.2.2) $h(X) \subseteq S(X)$, $f(X) \subseteq T(X)$, $g(X) \subseteq R(X)$,

(2.2.3) one of f(X), g(X) and h(X) is a G- complete subspace of X,

(2.2.4) the pairs (f, S), (g, T) and (h, R) are weakly compatible.

Then (a) one of the pairs (f, S), (g, T) and (h, R) has a coincidence point in X or

(b) S, T, R, f, g and h have a unique common fixed point in X.

The following example illustrates the Theorem 2.2.

Example 2.3 : Let $X = [0,\infty)$ and $G(x,y,z) = |x-y| + |y-z| + |z-x|$, $\forall\, x,y,z \in X$.

Let S,T,R,f,g,h : $X \to X$ be defined by $Sx = \frac{x}{2}$, $Tx = \frac{x}{4}$, $Rx = x$,

$fx = \frac{x}{16}$, $gx = \frac{x}{32}$, $hx = \frac{x}{8}$.

Clearly (2.2.2) – (2.2.4) are satisfied . Also $G(Sx, Ty, Rz) = 8\ G(fx,gy,hz)$ for all $x,y,z \in X$.

Clearly “0” is the unique common fixed point of S, T, R, f, g and h .

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