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SECOND-ORDER DUALITY FOR NONDIFFERENTIABLE MULTIOBJECTIVE PROGRAMMING INVOLVING (Φ, ρ)-UNIVEXITY

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ABSTRACT
The concepts of (Φ, ρ)-invexity have been given by Carsiti,Ferrara and Stefanescu[32]. We consider a second-order dual model associated to a multiobjective programming problem involving support functions and a weak duality result is established under appropriate second-order (Φ, ρ)-univexity conditions.

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1. INTRODUCTION
For nonlinear programming problems, a number of duals have been suggested among which the Wolfe dual [35,8] is well known. While studying duality under generalized convexity, Mond and Weir [36] proposed a number of deferent duals for nonlinear programming problems with nonnegative variables and proved various duality theorems under appropriate pseudo-convexity/quasi-convexity assumptions.

The study of second order duality is significant due to the computational advantage over first order duality as it provides tighter bounds for the value of the objective function when approximations are used [10,16,24]. Mangasarian [12] considered a nonlinear programming problem and discussed second order duality under inclusion condition. Mond [14] was the first who present second order convexity. He also gave in [14] simpler conditions than Mangasarian using a generalized form of convexity. which was later called bonvexity by Bector and Chandra [2]. Further, Jeyakumar [37,30] and Yang [24] discussed also second order Mangasarian type dual formulation under ρ-convexity and generalized representation conditions respectively. In [20] Zhang and Mond established some duality theorems for second-order duality in nonlinear programming under generalized second-order B-invexity, defined in their paper. In [14] it was shown that second order duality can be useful from computational point of view, since one may obtain better lower bounds for the primal problem than otherwise. The case of some optimization problems that involve n-set functions was studied by Preda [38]. Recently, Yang *et al.*[24] proposed four second-order dual models for nonlinear programming problems and established some duality results under generalized second-order F -

convexity assumptions.
For $\Phi(x,a,(y,r)) = F(x,a;y) + rd^2(x,a)$, where $F(x,a;.)$ is sublinear on R^n, the definition of (Φ,ρ) - invexity reduces to the definition of (F,ρ)-convexity introduced by Preda[29], which in turn Jeyakumar[30] generalizes the concepts of F-convexity and ρ-invexity[31].

The more recent literature, Xu[21], Ojha [27], Ojha and Mukherjee [22] for duality under generalized (F,ρ)-convexity, Mishra [23] and Yang et al.[24] for duality under second order F-convexity. Liang *et al.*[25] and Hachimi[26] for optimality criteria and duality involving (F,α,ρ,d)-convexity or generalized $\{F,\alpha,\rho,d\}$-type functions.The (F,ρ)-convexity was recently generalized to (Φ,ρ)-invexity by Caristi , Ferrara and Stefanescu [32],and here we will use this concept to extend some theoretical results of multiobjective programming.

Whenever the objective function and all active restriction functions satisfy simultaneously the same generalized invexity at a Kuhn-Tucker point which is an optimum condition, then all these functions should satisfy the usual invexity, too. This is not the case in multiobjective programming ; Ferrara and Stefanescu[28] showed that sufficiency Kuhn-Tucker condition can be proved under (Φ,ρ)-invexity, even if Hanson's invexity is not satisfied, Puglisi[34].The interested reader may consult[1,3,4,5,6,7,9,11,13,15,17,18,19,33].
Therefore, the results of this paper are real extensions of the similar results known in the literature.

In Section 2 we define the second-order (Φ, ρ)-univexity . In Section 3 we consider a class of Multiobjective programming problems and for the dual model we prove a weak duality result.

2. NOTATIONS AND PRELIMINARIES

we denote by R^n the n-dimensional Euclidean space, and by R^n_+ its nonnegative orthant .
Further, $R^n_+ = \{x \in R \mid x > 0\}$ .For any vector $x \in R^n$, $y \in R^n$, we denote $x^T y = \sum_{i=1}^{n} x_i y_i$ .Let $C \subset R^n$ be a compact convex set . The support function of C is defined by $s(x|C) = \max\{x^T y \mid y \in C\}$.Being convex and every where finite, it has a subdiferential , that is, there exist $z \in R^n$ such that $s(y|C) \ge s(x|C) + z^T(y-x)$ for all $y \in C$.
The subdifferantials of $s(x|C)$ is given by $\partial s(x|C) = \{z \in C \mid z^T x = s(x|C)\}$.
For any set $D \subset R^n$, the normal cone to D at a point $x \in D$ is defined by
$N_D(x) = \{y \in R^n \mid y^T(z-x) \le 0, \text{ for all } z \in D\}$.

For a compact convex set C we obviously have $y \in N_C(x)$ if and only if $s(y|C) = x^T y$, or equivalently, if $x \in \partial s(y|C)$.

We consider $f : R^n \to R^p$, $g : R^n \to R^q$,are differential functions and $X \subset R^n$ is an open set. We define the following multiobjective programming problem:

(P) minimize $f(x) = \left(f_1(x) .... f_p(x) \right)$

$\quad$ $subject\ to\ g(x) \geq 0x, x \in X$

Let X_0 be the set of all feasible solutions of (P) that is, $X_0 = \{x \in X \,|\, g(x) \geq 0\}$.

We quote some definitions and also give some new ones.

Definition 2.1

A vector $a \in X_0$ is said to be an efficient solution of problem (P) if there exit no $x \in X_0$ such that $f(a) - f(x) \in R_+^p \setminus \{0\}$ i.e., $f_i(x) \leq f_i(a)$ for all $i \in \{1,.,.,., p\}$, and for at least one $j \in \{1,.,.,., p\}$ we have $f_i(x) < f_i(a)$.

Definition 2.2

A point $a \in X_0$ is said to be a weak efficient solution of problem (VP) if there is no $x \in X$ such that $f(x) < f(a)$.

Definition 2.3

A point $a \in X_0$ is said to be a properly efficient solution of (VP) if it is efficient and there exist a positive constant K such that for each $x \in X_0$ and for each $i \in \{1, 2......p\}\ satisfying$ $f_i(x) < f_i(a)$, there exist at least one $i \in \{1, 2......p\}\ such\ that$ $f_j(a) < f_j($ and $f_i(a) - f_i(x) \leq K\left(f_j(x) - f_j(a) \right)$.

Denoting by WE(P), E(P) and PE(P) the sets of all weakly efficient, efficient and properly efficient solutions of (VP), we have $\mathrm{PE(P)} \subseteq \mathrm{E(P)} \subseteq \mathrm{WE(P)}$.

We denote by $\nabla f(a)$ the gradient vector at a of a differentiable function $f : R^p \to R$, and by $\nabla^2 f(a)$ the Hessian matrix of f at a. For a real valued twice differentiable function $\psi(x, y)$ defined on an open set in $R^p \times R^q$, we denote by $\nabla_x \psi(a,b)$ the gradient vector of ψ with respect to x at (a,b), and by $\nabla_{xx}\psi(a,b)$ the Hessian matrix with respect to x at (a,b). Similarly, we may define $\nabla_y \psi(a,b)$, $\nabla_{xy}\psi(a,b)$ and $\nabla_{yy}\psi(a,b)$.

For convenience, let us write the definitions of (Φ, ρ)-univexity from[1], Let $\varphi : X_0 \to R$ be a differentiable function $(X_0 \subseteq R^n)$, $X \subseteq X_0$, and $a \in X_0$. An element of all (n+1)-dimensional Euclidean Space R^{n+1} is represented as the ordered pair (z, r) with $z \in R^n$ and $r \in R, \rho$ is a real number and Φ is real valued function defined on $X_0 \times X_0 \times R^{n+1}, such\ that\ \Phi(x, a, .) is$ convex on R^{n+1} and $\Phi(x, a, (0, r)) \geq 0,$ for every

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$(x,a)\in X_0\times X_0$ and $r\in R_+$. $b_0,b_1 : X\times X\times[0,1]\to R_+$ $b(x,a)=\lim_{\lambda\to 0} b(x,a,\lambda)\geq$, and b does not depend upon λ if the corresponding functions are differentiable. $\psi_0,\psi_1 : R\to R$ is an n-dimensional vector- valued function and $h: X\times R^n\to R$ be differentiable function.

We assume that $\psi_0,\psi_1 : R\to R$ satisfying $u\leq 0\Rightarrow \psi_0(u)\leq 0$ *and* $u\leqq 0\Rightarrow \psi_1(u)\leqq 0,$ *and* $b_0(x,a)>0$ *and* $b_1(x,a)\geqq 0.$ and $\psi_0(\alpha)=-\psi_0(\alpha)$ and $\psi_1(-\alpha)=-\psi_1(\alpha)$.

Example 2.1

$$\min f(x)=x-1$$
$$g(x)=-x-1\leq 0, x\in X_0\in[1,\infty)$$
$$\Phi(x,a;(y,r))=2(2^r-1)|x-a|+\langle y,x-a\rangle$$

for $\psi_0(x)=x$, $\psi_1(x)=-x$, $\rho_1=\frac{1}{2}$(for f) and $\rho=1$(for g), then this is (ϕ,ρ)-univex but it is not (ϕ,ρ)-invex .

Definition 2.4

A real-valued twice differentiable function $f(.,y): X\times X\to R$ is said to be second-order (Φ,ρ)-univex at $u\in X$ with respect to $p\in R^n$, if for all $b: X\times X\to R_+$, $\Phi: X\times X\times R^{n+1}\to R$, ρ is a real number, we have

$$b(x,u)[\psi\{f_i(x,y)-f_i(u,y)+\frac{1}{2}p^T\nabla^2 f_i(u,y)p\}] \geqq \Phi(x,u;(\nabla f_i(u,y)+\nabla^2 f_i(u,y)p,\rho_i)) \tag{2.1}$$

Definition 2.5

A real-valued twice differentiable function $f(.,y): X\times X\to R$ is said to be second-order (Φ,ρ)-pseudounivex at $a\in X$ with respect to $p\in R^n$,if for all $b: X\times X\to R_+$, $\Phi: X\times X\times R^{n+1}\to R$, ρ is a real number, we have

$$\Phi(x,u;(\nabla f_i(u,y)+\nabla^2 f_i(u,y)p,\rho_i))\geqq 0 \Rightarrow b(x,u)[\psi\{f_i(x,y)-f_i(u,y)+\frac{1}{2}p^T\nabla^2 f_i(u,y)p\}]\geqq 0 \tag{2.2}$$

Definition 2.6

A real-valued twice differentiable function $f(.,y): X\times X\to R$ is said to be second-order (Φ,ρ)-quasiunivex at $a\in X$ with respect to $p\in R^n$,if for all $b: X\times X\to R_+$, $\Phi: X\times X\times R^{n+1}\to R$, ρ is a real number, we have

$$b(x,u)[\psi\{f_i(x,y)-f_i(u,y)+\frac{1}{2}p^T\nabla^2 f_i(u,y)p\}]\leqq 0 \Rightarrow \Phi(x,u;(\nabla f_i(u,y)+\nabla^2 f_i(u,y),\rho_i))\leqq 0 \tag{2.3}$$

Remark 2.1

(i) If we consider the case $b\equiv 1$, $\Phi(x,u;(\nabla f(u),\rho))=F(x,u;\nabla f(u))$ (with F is sublinear in third argument, then the above definition reduce to Definition 4 of Chen[4] .

(ii)When $h(u,y) = y^T \nabla_{uu} f(u) y/2$ and $\Phi(x,u;(\nabla f(u),\rho)) = F(x,u;\nabla f(u)) = \eta(x,u)^T \nabla f(u)$ where $\eta : X \times X \to R^n$, the above definition reduce to η -(pseudo/quasi)-bonvexity.

Example 2.1

We present here a function which is second-order (Φ,ρ) -univex for $b \geqq 1$. Let us consider $X = (0,\infty)$ and
$f : X \to R$, $f(x) = x \log x$, $h : X \times R \to R, h(u,y) = -y \log u$. We have
$\nabla_u f(u) = 1 + \log u, \nabla_{uu} f(u) = \frac{1}{u}, \nabla_y h(u,y) = -\log u$, $\Phi : X \times X \times R^{n+1} \to R$, taking
$\rho = 0$ $\Phi(x,y;b) = |b| + |b|^2$

It is obvious our mapping is more generalized rather than previous ones.

Hence $f(x) = x \log x$ is second-order (Φ,ρ) -univex at $u \in X$, with respect to $h(u,y) = -y \log u$.

A real valued twice differentiable function g is second order F-pseudoconcave if $-g$ is second order F-pseudoconvex.

We shall make use of the following generalized Schwartz inequality:

$x^T Ay \leqq (x^T Ax)^{\frac{1}{2}} (y^T Ay)^{\frac{1}{2}}$, where $x, y \in R^n$ and $A \in R^n \times R^n$ is a positive semidefinite matrix. Equality holds if for some $\lambda \geqq 0, Ax = \lambda Ay$.

3. MOND-WEIR TYPE SECOND ORDER SYMMETRIC DUALITY

We consider here the following pair of second order nondifferentiable multiobjective with $r-$objectives and establish weak, strong and converse duality theorems.

(MP)

minimize

$H(x,y,w,p) = \{H_1(x,y,w,p), H_2(x,y,w,p),.,.,H_r(x,y,w,p)\}$

subject to

$$\sum_{i=1}^{r} \lambda_i [\nabla_y f_i(x,y) - C_i w_i + \nabla_{yy} f_i(x,y) p_i)] \leqq 0 \tag{3.1}$$

$$y^T \sum_{i=1}^{r} \lambda_i [\nabla_y f_i(x,y) - C_i w_i + \nabla_{yy} f_i(x,y) p_i)] \geqq 0 \tag{3.2}$$

$$w_i^T C_i w_i \leqq 1, i = 1,2,.,.,r \tag{3.3}$$

$$\lambda > 0 \tag{3.4}$$

$$x \geqq 0 \tag{3.5}$$

(MD)

maximize

$J(u,v,a,q) = \{J_1(u,v,a,q), J_2(u,v,a,q),.,.,J_r(u,v,a,q)\}$

subject to

$$\sum_{i=1}^{r}\lambda_i[\nabla_x f_i(u,v)+E_i a_i+\nabla_{xx} f_i(u,v)q_i)] \geqq 0 \qquad (3.6)$$

$$u^T\sum_{i=1}^{r}\lambda_i[\nabla_x f_i(u,v)+E_i a_i+\nabla_{xx} f_i(u,v)q_i)] \leqq 0 \qquad (3.7)$$

$$a_i^T E_i a_i \leqq 1, i=1,2,.,.,r \qquad (3.8)$$

$$\lambda > 0 \qquad (3.9)$$

$$v \geqq 0 \qquad (3.10)$$

Where $H_i(x,y,w,p)= f_i(x,y)+(x^T E_i x)^{\frac{1}{2}} - y^T C_i w_i - \frac{1}{2} p_i^T \nabla_{yy} f_i(x,y) p_i$

$J_i(u,v,a,q)= f_i(u,v)-(v^T C_i v)^{\frac{1}{2}} + u^T E_i a_i - \frac{1}{2} q_i^T \nabla_{xx} f_i(u,v) q_i$

$\lambda_i \in R, p_i \in R^n, q_i \in R^n, i=1,2,.,.,r$ and $f_i, i=1,2,.,.,r$ are thrice differentiable function from $R^n \times R^n \to R$, E_i and $C_i, i=1,2,.,.,r$ are positive semidefinite matrices. Also, we mean here, $b_i = R^n \times R^m \times R^n \times R^m \to R_+$ $p=(p_1,p_2,.,.,p_r)$, $q=(q_1,q_2,.,.,q_r)$, $w=(w_1,w_2,.,.,w_r)$, $a=(a_1,a_2,.,.,a_r)$

Remark: 3.1

Since the objective functions of (MP) and (MD) contain the support functions $s(x|C_i)$ and $s(v|D_i)$, $i=1,2,.,.,p$, these problems are nondifferentiable multiobjective programming problems.

Theorem 3.1 (Weak duality)

Let (x,y,λ,w,p) be a feasible solution of (MP) and (u,v,λ,a,q) a feasible solution of (MD). Then the inequalities can not hold simultaneously:

(i) $\sum_{i=1}^{r}\lambda_i[f_i(.,v)+(.)^T E_i a_i]$ is second order (Φ,ρ)-pseudounivex at u,

(ii) $\sum_{i=1}^{r}\lambda_i[f_i(x,.)-(.)^T C_i w_i]$ is second order (Φ,ρ)-pseudounicave at y

(iii) $\Phi(x,u;(\xi,\rho))+u^T\xi \geqq 0$, for $\xi \in R^n$, and

(iv) $\Phi(v,y;(\zeta,\rho))+y^T\zeta \geqq 0$, for $\zeta \in R^n$, then

$H(x,y,w,p) \not\leq J(u,v,a,q)$.

Proof

With the help of $\sum_{i=1}^{r}\lambda_i[\nabla_x f_i(u,v)+E_i a_i+\nabla_{xx} f_i(u,v)q_i)]$, we have

$$\Phi(x,u;(\sum_{i=1}^{r}\lambda_i[\nabla_x f_i(u,v)+E_i a_i+\nabla_{xx} f_i(u,v)q_i)],\rho_i))$$
$$+u^T\sum_{i=1}^{p}\lambda_i\{\nabla_u f_i(u,v)+w_i+\nabla_{\mu} g_i(u,v,\mu_i)\}\geqq 0$$

(By hypothesis (iii) and (3.7), also by the second order (Φ,ρ)-pseudounivexity of $\sum_{i=1}^{r}\lambda_i[f_i(.,v)+(.)^T E_i a_i]$ at u, with property of b and ψ, provides

$$\sum_{i=1}^{r}\lambda_i[f_i(x,v)+(x)^T E_i a_i]\geqq \sum_{i=1}^{r}\lambda_i(f_i(u,v)+u^T E_i a_i-\frac{1}{2}q_i^T\nabla_{xx} f_i(u,v)q_i) \tag{3.11}$$

Now, $\zeta=-\sum_{i=1}^{r}\lambda_i[\nabla_y f_i(x,y)-C_i w_i+\nabla_{yy} f_i(x,y)p_i)]$, we have

$\Phi(v,y;(\zeta,\rho))+y^T\zeta\geqq 0$ (by hypothesis (iv),(3.2) and by the second order (Φ,ρ) pseudounicavity $\sum_{i=1}^{r}\lambda_i[f_i(x,.)-(.)^T C_i w_i]$ at y, with property of b and ψ, gives

$$\sum_{i=1}^{r}\lambda_i[f_i(x,v)-(v)^T C_i w_i]\leqq \sum_{i=1}^{r}\lambda_i[f_i(x,y)-y^T C_i w_i-\frac{1}{2}p_i^T\nabla_{yy} f_i(x,y)p_i] \tag{3.12}$$

Combining (3.11) and (3.12), we get

$$\sum_{i=1}^{r}\lambda_i[(x)^T E_i a_i+v^T C_i w_i]\geqq$$
$$\sum_{i=1}^{r}\lambda_i[\{(f_i(u,v)+u^T E_i a_i-\frac{1}{2}q_i^T\nabla_{xx} f_i(u,v)q_i)\}-\{f_i(x,y)+y^T C_i w_i+\frac{1}{2}p_i^T\nabla_{yy} f_i(x,y)p_i\}]$$

Applying Schwartz inequality, (3.3) and (3.8), we get

$$\sum_{i=1}^{r}\lambda_i\{f_i(x,y)+(x^T E_i x)^{\frac{1}{2}}-y^T C_i w_i-\frac{1}{2}p_i^T\nabla_{yy} f_i(x,y)p_i\}$$
$$\geqq\sum_{i=1}^{r}\lambda_i\{(f_i(u,v)-(v^T C_i v)^{\frac{1}{2}}+u^T E_i a-\frac{1}{2}q_i^T\nabla_{xx} f_i(u,v)q_i)\}$$

Hence
$H(x,y,w,p)\not\leq J(u,v,a,q)$. †

Theorem 3.2 (Strong duality)

Let f be thrice differentiable on $R^n\times R^n$ and (x',y',λ',w',p') be a weak efficient solution for (MP), and $\lambda=\lambda'$,assume that

(i) $\nabla_{yy} f_i$ is nonsingular for all $i=1,2,.,.,r$;

(ii) the matrix $\sum_{i=1}^{r}\lambda_i'(\nabla_{yy} f_i p_i')_y$ is positive or negative definite, and ;

(iii) the set $[\nabla_y f_1 - C_1 w'_1 + \nabla_{yy} f_1 p'_1, \nabla_y f_2 - C_2 w'_2 + \nabla_{yy} f_2 p'_2, .,.,., \nabla_y f_r - C_r w'_r + \nabla_{yy} f_r p'_r\}$, are linearly independent;

where $f_i = f_i(x', y'), i = 1, 2, .,.,., r$. Then $(x', y', \lambda', a', q' = 0)$ is a feasible solution of (MD), $b_i(x', y', u', v') > 0, i = 1, 2, .,.,., r$, and the two objectives have the same values. Also, if the hypothesis of Theorem 3.1 are satisfied for all feasible solutions of (MP) and (MD), then $(x', y', \lambda', a', q' = 0)$ is an efficient solution for (MD).

Proof

Since $(x', y', \lambda', w', p')$ is a weak efficient solution of (MP), by Fritz-John condition [7],there exist $\alpha \in R^r$, $\beta \in R^n$, $\gamma \in R, v \in R^r$ and $\xi \in R^n$ such that

$$\sum_{i=1}^{r} \alpha_i [\nabla_x f_i + E_i a'_i - \frac{1}{2}(\nabla_{yy} f_i p'_i) x p'_i] + \sum_{i=1}^{r} \lambda'_i [\nabla_{yx} f_i + (\nabla_{yy} f_i p'_i) x](\beta - \gamma y') - \xi = 0 \tag{3.13}$$

$$\sum_{i=1}^{r} \alpha_i [\nabla_y f_i - C_i w'_i + \frac{1}{2}(\nabla_{yy} f_i p'_i)_y p'_i] + \sum_{i=1}^{r} \lambda'_i [\nabla_{yy} f_i + (\nabla_{yy} f_i p'_i)_y](\beta - \gamma y')$$

$$-\gamma \sum_{i=1}^{r} \lambda'_i [\nabla_y f_i - C_i w'_i + (\nabla_{yy} f_i p'_i)] = 0 \tag{3.14}$$

$$(\beta - \gamma y')^T [\nabla_y f_i - C_i w'_i + \nabla_{yy} f_i p'_i] - \delta_i = 0, i = 1, 2, .,.,., r \tag{3.15}$$

$$\alpha_i C_i y' + (\beta - \gamma y')^T \lambda'_i C_i = 2 v_i C_i w'_i, i = 1, 2, .,.,., r \tag{3.16}$$

$$[(\beta - \gamma y')\lambda'_i - \alpha_i p'_i]^T \nabla_{yy} f_i = 0, i = 1, 2, .,.,., r \tag{3.17}$$

$$x'^T E_i a'_i = (x'^T E_i x'_i)^{\frac{1}{2}}, i = 1, 2, .,.,., r \tag{3.18}$$

$$\beta^T \sum_{i=1}^{r} \lambda'_i [\nabla_y f_i - C_i w'_i + \nabla_{yy} f_i p'_i] = 0 \tag{3.19}$$

$$\gamma y' \sum_{i=1}^{r} \lambda'_i [\nabla_y f_i - C_i w'_i + \nabla_{yy} f_i p'_i] = 0 \tag{3.20}$$

$$v_i (w'^T_i C_i w'_i - 1) = 0, i = 1, 2, .,.,., r \tag{3.21}$$

$$\delta^T \lambda' = 0 \tag{3.22}$$

$$x'^T \xi = 0 \tag{3.23}$$

$$a'^T_i E_i a'_i \leqq 1, i = 1, 2, .,.,., r \tag{3.24}$$

$$(\alpha, \beta, \gamma, v, \delta, \xi) \geqq 0 \tag{3.25}$$

$$(\alpha, \beta, \gamma, v, \delta, \xi) \neq 0 \tag{3.26}$$

Since $\lambda' > 0$ and $\delta \geqq 0$, (3.22) implies $\delta = 0$. Consequently, (3.15) gives

$$(\beta - \gamma y')^T [\nabla_y f_i - C_i w'_i + \nabla_{yy} f_i p'_i] = 0 \tag{3.27}$$

Since $\nabla_{yy} f_i$ is nonsingular for $i = 1, 2, .,.,., r$, from (3.17), it follows that

$$(\beta - \gamma y')\lambda'_i = \alpha_i p'_i, i = 1, 2, .,.,., r. \tag{3.28}$$

from (3.14), we get $\sum_{i=1}^{r}(\alpha_i-\gamma\lambda_i')(\nabla_y f_i - C_i w_i') + \sum_{i=1}^{r}\lambda_i'\nabla_{yy} f_i(\beta-\gamma y'-\gamma p_i')$

$$+\sum_{i=1}^{r}(\nabla_{yy} f_i p_i')_y[(\beta-\gamma y')\lambda_i' - \frac{1}{2}\alpha_i p_i'] = 0$$

using (3.28), we get

$$\sum_{i=1}^{r}(\alpha_i-\gamma\lambda_i')(\nabla_y f_i - C_i w_i' + \nabla_{yy} f_i p_i') + \frac{1}{2}\sum_{i=1}^{r}\lambda_i'(\nabla_{yy} f_i p_i')_y(\beta-\gamma y') = 0 \qquad (3.29)$$

Premultiplying (3.29) by $(\beta-\gamma y')^T$ and using (3.27), we get

$(\beta-\gamma y')^T\sum_{i=1}^{r}\lambda_i'(\nabla_{yy} f_i p_i')_y(\beta-\gamma y') = 0$, by hypothesis (ii) implies

$$\beta = \gamma y' \qquad (3.30)$$

Therefore, from (3.29), we get $\sum_{i=1}^{r}(\alpha_i-\gamma\lambda_i')(\nabla_y f_i - C_i w_i' + \nabla_{yy} f_i p_i') = 0$, which by hypothesis (iii) gives $\alpha_i = \gamma\lambda_i', i = 1,2,.,.,r$ (3.31)

If $\gamma = 0$, then $\alpha_i = 0, 1 = 1,2,.,.,r$ and from (3.30), $\beta = 0$. Also from (3.13) and (3.16), we get, $\xi_i = 0, v_i = 0, i = 1,2,.,.,r$. Thus $(\alpha,\beta,\gamma,v,\delta,\xi) =\ 0$, a contradiction to (3.26). Hence $\gamma > 0$, since $\lambda_i' > 0, i = 1,2,.,.,r$, (3.31) implies $\alpha_i > 0, 1 = 1,2,.,.,r$. Using (3.30) in (3.28), $\alpha_i p_i' = 0, i = 1,2,.,.,r$, hence $p_i' = 0, i = 1,2,.,.,r$. Using (3.30) and $p_i' = 0, i = 1,2,.,.,r$ in (3.13), it gives $\sum_{i=1}^{r}\alpha_i[\nabla_x f_i + E_i a_i'] = \xi$, which by (3.31) gives $\sum_{i=1}^{r}\lambda_i'[\nabla_x f_i + E_i a_i'] = \frac{\xi}{\gamma} \geqq 0$ (3.32)

$$x'^T\sum_{i=1}^{r}\lambda_i'[\nabla_x f_i + E_i a_i'] = x'^T\frac{\xi}{\gamma} = 0 \qquad (3.33)$$

Also, from (3.30), we get

$$y' = \frac{\beta}{\gamma} \geqq 0 \qquad (3.34)$$

Hence from (3.24) and (3.32-3.34), $(x', y', \lambda', a', q' = 0)$ is feasible for (MD).

Let $2\frac{v_i}{\alpha_i} = t$. Then $t \geqq 0$ and from (3.16) and (3.30) $C_i y' = tC_i w_i'$ (3.35)

Which is the condition in the Schwartz inequality. Therefore

$$y'^T C_i w_i' = (y'^T C_i y')^{\frac{1}{2}}(w_i'^T C_i w_i')^{\frac{1}{2}}.$$

In case, $v_i > 0$, (3.21) gives $w_i'^T C_i w_i' = 1$ and so $y'^T C_i w_i' = (y'^T C_i y')^{\frac{1}{2}}$. In case $v_i = 0$, (3.35) gives $C_i y' = 0$ and so $y'^T C_i w_i' = (y'^T C_i y')^{\frac{1}{2}} = 0$.

Thus in either case $y'^T C_i w'_i = (y'^T C_i y')^{\frac{1}{2}}$. (3.36)

Hence $H_i(x', y', w', p' = 0) = f_i(x', y') + (x'^T E_i x')^{\frac{1}{2}} - y'^T C_i w_i$

$= f_i(x', y') - (y'^T C_i y')^{\frac{1}{2}} + x'^T E_i a'_i = J_i(x', y', a', q' = 0)$ (using (3.18) and (3.36)).

Now follows from Theorem 3.1 that $(x', y', \lambda', a', q' = 0)$ is an efficient solution for (MD).

†

A converse duality theorem may be merely stated as its proof would run analogously to that of Theorem 3.2.

Theorem 3.3 (Converse duality)

Let f be thrice differentiable on $R^n \times R^n$ and $(u', v', \lambda', a', q')$ be a weak efficient solution for (MD), and $\lambda = \lambda'$ fixed in (MP) .Assume that

(i) $\nabla_{xx} f_i$ is nonsingular for all $i = 1, 2, ., ., r$ **;**

(ii) the matrix $\sum_{i=1}^{r} \lambda'_i (\nabla_{xx} f_i q'_i)_x$ is positive or negative definite, and ;

(iii) the set $[\nabla_x f_1 + E_1 a'_1 + \nabla_{xx} f_1 q'_1, \nabla_x f_2 + E_2 a'_2 + \nabla_{xx} f_2 q'_2, ., ., \nabla_x f_r + E_r a'_r + \nabla_{xx} f_r q'_r\}$, are linearly independent;

where $f_i = f_i(u', v'), i = 1, 2, ., ., r$. Then $(u', v', \lambda', w', p' = 0)$ is a feasible solution of (MP), $b_i(x', y', u', v') > 0, i = 1, 2, ., ., r$,and the two objectives have the same values. Also, if the hypothesis of Theorem 3.1 are satisfied for all feasible solutions of (MP) and (MD), then $(u', v', \lambda', w', p' = 0)$ is an efficient solution for (MP). †

4. SPECIAL CASES

(i) If $b = 1, \psi \equiv I$, $E_i = C_i = 0, i = 1, 2, ., ., r$, and $\Phi(x, u; (\nabla f(u), \rho)) = F(x, u; \nabla f(u))$ for $\rho = 0$ then (MP) and (MD) reduce to the second order multiobjective symmetric dual programstudied by Suneja *et al.*[16] with omission of non-negativity constraints from (MP) and (MD). If in addition $p = q = 0$, and $r = 1$, then we get the first order symmetric dual programs of Chandra et al.[4].

(ii) If $b = 1, \psi \equiv I$,we set $p = q = 0$, and $\Phi(x, u; (\nabla f(u), \rho)) = F(x, u; \nabla f(u))$ for $\rho = 0$ in (MP) and (MD), then we obtain a pair of first order symmetric dual nondifferentiable multiobjective programs considered by Mond et al.[15].

(iii) If we set, $b = 1, \psi \equiv I$, $\Phi(x, u; (\nabla f(u), \rho)) = F(x, u; \nabla f(u))$ for $\rho = 0$ in (MP) and (MD), then we obtain a pair of second order symmetric dual nondifferentiable multiobjective programs considered by Ahmad et al.[20].

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