KATHMANDU UNIVERSITY JOURNAL OF SCIENCE, ENGINEERING AND TECHNOLOGY
VOL. 7, No. I, SEPTEMBER, 2011, pp 48- 55

EXISTENCE AND UNIQUENESS OF CONTINUOUS SOLUTION OF MIXED TYPE INTEGRAL EQUATIONS IN CONE METRIC SPACE

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Received 7 November, 2009; Revised 16 August, 2010

ABSTRACT
In this paper we investigate the existence and uniqueness for Volterra-Fredholm type integral equations in cone metric spaces. The result is obtained by using the some extensions of Banach's contraction principle in complete cone metric space.

Mathematics Subject Classification: 45N05, 47G20, 34K05, 47H10.
Keywords: Cone metric space, Contractive mapping, ordered Banach space.

INTRODUCTION
The purpose of this paper is study the existence and uniqueness of solutions for the Volterra-Fredholm type integral and integrodifferential equations of the forms:

$$x(t) = f(t) + \int_0^t k(t,s,x(s))ds + \int_0^b h(t,s,x(s))ds, \quad t \in J = [0,b], \tag{1}$$

and

$$x^{'}(t) = f(t) + \int_0^t k(t,s,x(s))ds + \int_0^b h(t,s,x(s))ds, \quad t \in J = [0,b], \tag{2}$$

$$x(0) = x_0, \tag{3}$$

where $f : J \to Z,\ k, h : J \times J \times Z \to Z$ are continuous and the given x_0 is element of $Z,\ Z$ is a Banach space with norm $\|\cdot\|$.

Many authors have been studied the problems of existence, uniqueness, continuation and other properties of solutions of these type or special forms of the equations (1) and $(2)-(3)$ are studied by different techniques, for example, see $[3, 6, 8, 9, 10, 11, 12, 13, 14]$ and the references given therein.

The objective of the present paper is to study the existence and uniqueness of solution of the system (1) and the system $(2)-(3)$ under the conditions in respect of the cone metric space and fixed point theory. Hence we extend and improve some results reported in $[2, 6, 9, 12, 14, 15]$. We are motivated by the work of P. Raja and S. M. Vaezpour in $[15]$

and influenced by the work of B. G. Pachpatte[12]. The paper is organized as follows: Section 2, we present the preliminaries and the statement of our results. In Section 3 deals with proof of theorems. Finally in Section 4, we give examples to illustrate the applications of our results.

PRELIMINARIES AND STATEMENT OF RESULTS

Let us recall the concepts of the cone metric space and we refer the reader to $[1, 4, 5, 7]$ for the more details. Let E be a real Banach space and P is a subset of E. Then P is called a cone if and only if,

1. P is closed, nonempty and $P \neq \{0\}$;
2. $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;
3. $x \in P$ and $-x \in P \Rightarrow x = 0$.

For a given cone $P \subset E$, we define a partial ordering relation $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$, for every $x, y \in E$. The least positive number satisfying above is called the normal constant of P.

In the following way, we always suppose E is a real Banach space, P is a cone in E with $intP \neq \phi$, and $\leq$ is partial ordering with respect to P.

Definition 1: Let Xbe a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d_1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d_2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space. The concept of cone metric space is more general than that of metric space. The following example is a cone metric space, see[15].

Example 1: Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \mathbb{R}$, and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant, and then (X, d) is a cone metric space.

Definition 2: Let X be an ordered space. A function $\Phi : X \to X$ is said to a comparison function if for every $x, y \in X$, $x \leq y$, implies that $\Phi(x) \leq \Phi(y)$, $\Phi(x) \leq x$ and $\lim_{n\to\infty} \|\Phi^n(x)\| = 0$, for every $x \in X$.

Example 2: Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$. It is easy to check that $\Phi : E \to E$, with $\Phi(x, y) = (ax, ay)$, for some $a \in (0, 1)$ is a comparison function. Also if Φ_1, Φ_2 are two comparison functions over $\mathbb{R}$, then $\Phi(x, y) = (\Phi_1(x), \Phi_2(y))$ is also a comparison function over E.

Let $B = C([0, b], Z)$ be the Banach space of all continuous functions from $[0, b]$ into Z endowed with supremum norm

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in [0, b]\}.$$

Let $P = \{(x, y) : x, y \geq 0\} \subset E = \mathbb{R}^2$, and define $d(f, g) = (\|f - g\|_\infty, \alpha\|f - g\|_\infty)$, for every $f, g \in B$. Then it is easily seen that (B, d) is a cone metric space.

Definition 3: The function $x \in B$ given by

$$x(t) = x_0 + \int_0^t f(s)ds + \int_0^t [\int_0^s k(s, \tau, x(\tau))d\tau + \int_0^b k(s, \tau, x(\tau))d\tau]ds, \quad t \in J$$

is called the solution of the initial value problem $(2) - (3)$.

We need the following theorem for further discussion:

Lemma 1: Let (X, d) be a complete cone metric space, where P is a normal cone with normal constant K. Let $f : X \to X$ be a function such that there exists a comparison function $\Phi : P \to P$ such that

$$d(f(x),f(y)) \leq \Phi(d(x, y)),$$

for every $x, y \in X$. Then f has a unique fixed point.

We list the following hypotheses for our convenience:

(H_1) There exist continuous functions $p_1, p_2 : J \times J \to \mathbb{R}^+$ and a comparison function $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$(\|k(t,s,u) - k(t,s,v)\|, \alpha\|k(t,s,u) - k(t,s,v)\|) \leq p_1(t,s)\Phi(d(u,v)),$$

and

$$(\|h(t,s,u) - h(t,s,v)\|, \alpha\|h(t,s,u) - h(t,s,v)\|) \leq p_2(t,s)\Phi(d(u,v)),$$

for every $t, s \in J$ and $u, v \in Z$.

(H_2)

$$\sup_{t \in J} \int_0^b [p_1(t,s) + p_2(t,s)]ds = 1.$$

(H_3)

$$\int_0^b \int_0^b [p_1(t,s) + p_2(t,s)]dsdt \leq 1.$$

Our main results are given in the following theorems:

Theorem 1: Assume that hypotheses $(H_1) - (H_2)$ hold. Then the integral equation (1) has a unique solution x on J.

Theorem 2: Assume that hypotheses (H_1) and (H_3) hold. Then the initial value problem $(2) - (3)$ has a unique solution x on J.

PROOF OF THEOREMS

The operator $F : B \to B$ is defined by

$$Fx(t) = f(t) + \int_0^t k(t,s,x(s))ds + \int_0^b h(t,s,x(s))ds, \quad t \in J. \tag{4}$$

By using the hypotheses $(H_1) - (H_2)$, we have

$$
\begin{aligned}
&(\|Fx(t)-Fy(t)\|,\alpha\|Fx(t)-Fy(t)\|)\\
&\leq (\|\int_0^t k(t,s,x(s))ds+\int_0^b h(t,s,x(s))ds-\int_0^t k(t,s,y(s))ds-\int_0^b h(t,s,y(s))ds\|,\\
&\quad \alpha\|\int_0^t k(t,s,x(s))ds+\int_0^b h(t,s,x(s))ds-\int_0^t k(t,s,y(s))ds-\int_0^b h(t,s,y(s))ds\|)\\
&\leq (\int_0^t \|k(t,s,x(s))-k(t,s,y(s))\|ds+\int_0^b \|h(t,s,x(s))-h(t,s,y(s))\|ds,\\
&\quad \alpha\int_0^t \|k(t,s,x(s))-k(t,s,y(s))\|ds+\alpha\int_0^b \|h(t,s,x(s))-h(t,s,y(s))\|ds)\\
&\leq (\int_0^t \|k(t,s,x(s))-k(t,s,y(s))\|ds,\alpha\int_0^t \|k(t,s,x(s))-k(t,s,y(s))\|ds)\\
&\quad +(\int_0^b \|h(t,s,x(s))-h(t,s,y(s))\|ds,\alpha\int_0^b \|h(t,s,x(s))-h(t,s,y(s))\|ds)\\
&\leq \int_0^t p_1(t,s)\Phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)ds+\int_0^b p_2(t,s)\Phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)ds\\
&\leq \int_0^b p_1(t,s)\Phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)ds+\int_0^b p_2(t,s)\Phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)ds\\
&\leq \int_0^b [p_1(t,s)+p_2(t,s)]\Phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)ds\\
&\leq \Phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)\int_0^b [p_1(t,s)+p_2(t,s)]ds\\
&=\Phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty), \qquad (5)
\end{aligned}
$$

for every $x,y\in B$. This implies that $d(Fx,Fy)\leq \Phi(d(x,y))$, for every $x,\ y\in B$. Now an application of Lemma 1, the operator has a unique point in B. This means that the equation (1) has unique solution. This completes the proof of the Theorem 1.

We want to prove that the operator $G:B\to B$ is defined by

$$Gx(t)=x_0+\int_0^t f(s)ds+\int_0^t[\int_0^s k(s,\tau,x(\tau))d\tau+\int_0^b k(s,\tau,x(\tau))d\tau]ds, \quad t\in J \tag{6}$$

has unique fixed point. This fixed point is then a solution of equations$(2)-(3)$. This can be proved by using the hypotheses and with suitable modifications, and closely looking at proof of Theorem 1. We omit the details here. This completes the proof of the Theorem 2.

APPLICATION

In this section we give examples to illustrate the usefulness of our results. In equations, (1) and$(2)-(3)$, we define:

$$k(t,s,x) = ts + \frac{xs}{2}, \quad h(t,s,x) = (ts)^2 + \frac{tsx^2}{2}, \quad s,t \in [0,1], \quad x \in C([0,1],\mathbb{R}), \tag{7}$$

and metric $d(x,y) = (\|x-y\|_\infty, \alpha\|x-y\|_\infty)$ on $C([0,1],\mathbb{R})$ and $\alpha \geq 0$. Then clearly $C([0,1],\mathbb{R})$ is a complete cone metric space.

Now we have

$$\begin{aligned}
&(|k(t,s,x(s)) - k(t,s,y(s))|, \alpha|k(t,s,x(s)) - k(t,s,y(s))|) \\
&= (|ts + \frac{xs}{2} - ts - \frac{ys}{2}|, \alpha|ts + \frac{xs}{2} - ts - \frac{ys}{2}|) \\
&= (|\frac{xs}{2} - \frac{ys}{2}|, \alpha|\frac{xs}{2} - \frac{ys}{2}|) \\
&= (\frac{s}{2}|x-y|, \alpha\frac{s}{2}|x-y|) \\
&= \frac{s}{2}(|x-y|, \alpha|x-y|) \\
&\leq \frac{s}{2}(\|x-y\|_\infty, \alpha\|x-y\|_\infty) \\
&= p_1^*\Phi^*(\|x-y\|_\infty, \alpha\|x-y\|_\infty),
\end{aligned} \tag{8}$$

where $p_1^*(t,s) = s,$ which is continuous function of $[0,1]\times[0,1]$ into $\mathbb{R}^+$ and a comparison function $\Phi^* : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Phi^*(x,y) = \frac{1}{2}(x,y).$ Similarly, we can show that $(|h(t,s,x(s)) - h(t,s,y(s))|, \alpha|h(t,s,x(s)) - h(t,s,y(s))|) \leq p_2^*\Phi^*(\|x-y\|_\infty, \alpha\|x-y\|_\infty),$

where $p_2^*(t,s) = st,$ which is continuous function of $[0,1]\times[0,1]$ into $\mathbb{R}^+.$

Moreover,

$$\int_0^1 [p_1^*(t,s) + p_2^*(t,s)]ds = \int_0^1 [s + st]ds = \frac{1}{2}(1+t)$$

And

$$\sup_{t\in[0,1]} \{1/2(1+t)\} = 1.$$

Also

$$\int_0^1 \int_0^1 [p_1^*(t,s) + p_2^*(t,s)]dsdt = \int_0^1 \int_0^1 [s+st]dsdt = \int_0^1 \frac{1}{2}(1+t)dt \leq \frac{3}{4} < 1.$$

With these choices of functions, all requirements of Theorem 2.2 and Theorem 2.3 are satisfied. Hence the existence and uniqueness are verified.

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