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APPROXIMATION OF FUNCTIONS BELONGING TO LIP (α, p) CLASS BY (E, 1)(N, p_n) MEANS OF ITS FOURIER SERIES

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ABSTRACT
The present paper deals with approximation of a function belonging to the Lip (α, p) class by product summability method. Here product of Euler (E,1) summability method and Nörlund (N, p_n) method has been taken. A new estimate on degree of approximation of a function f belonging to Lip (α, p) class has been determined by (E,1) (N, p_n) summability of a Fourier series.

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Definition

Let f be 2π-periodic function in $L^1(-\pi, \pi)$. The Fourier series associated with f at a point x is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{1}$$

Where

$$\left.\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \cos nt\ dt \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \sin nt\ dt \end{aligned}\right\} \text{ for } n = 1, 2, 3, \ldots,$$

and

$$a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, dt\, .$$

The L^p norm is defined by

$$\| f \|_p = \left\{ \int_0^{2\pi} | f(x) |^p \, dx \right\}^{\frac{1}{p}}, \; p \geq 1, \tag{2}$$

and the degree of approximation $E_n(f)$ is given by (Zygmund [12])

$$E_n(f) = \min_{T_n} \| f(x) - T_n(x) \|_p , \tag{3}$$

in term of n, where $T_n(x)$ is a trigonometric polynomial of degree n.

A function $f \in Lip\alpha$ if

$$|f(x+t) - f(x)| = O\left(|t|^{\alpha} \right) \quad \text{for} \quad 0 < \alpha \leq 1, \tag{4}$$

and $f(x) \in Lip(\alpha, p)$ for $0 \leq x \leq 2\pi$, if

$$\left(\int_0^{2\pi} | f(x+t) - f(x) |^p \, dx \right)^{\frac{1}{p}} = O\left(|t|^{\alpha} \right), \; 0 < \alpha \leq 1, \; p \geq 1 \text{ (McFadden [7]).} \tag{5}$$

If $p \to \infty$, Lip (α, p) class coincides with the Lip α class.

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with sequence of nth partial sum $s_n = \sum_{k=0}^{n} u_k$ and a sequence $\{p_n\}$ of real constant such that $P_n = \sum_{k=0}^{n} p_k \neq 0, \; P_{-1} = p_{-1} = 0.$

The Nörlund means of the sequence $\{s_n\}$ is given by (Hardy [4])

$$t_n^N = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} \, s_k . \tag{6}$$

If $t_n^N \to s$ as $n \to \infty$, then the sequence $\{s_n\}$ is summable to s by Nörlund method.

The Euler means (E,1) is given by (Hardy [4])

$$E_n^{(1)} = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} s_k . \tag{7}$$

If $E_n^{(1)} \to s$ as $n \to \infty$, then the sequence $\{s_n\}$ is summable to s by Euler method.

The (E,1) transform of the (N, p_n) transform define the (E,1)(N, p_n) transform of the partial sum $\{s_n\}$of series $\sum_{n=0}^{\infty} u_n$. The (E,1)(N, p_n) means defines a sequence $\{t_n^{EN}\}$ by

$$t_n^{EN} = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} t_k^N = = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{P_k} \sum_{r=0}^{k} p_{k-r} \, s_r \tag{8}$$

If $t_n^{EN} \to s$ as $n \to \infty$, then the sequence $\{s_n\}$ is said to summable by (E,1)(N, p_n) method to s.

Particular cases

Two particular cases of (E,1)(N, p_n) means are:

(i) $(E,1)(C,1)$ if $p_n = 1 \; \forall \; n$.

(ii) $(E,1)(C,\delta)$ if $p_n = \binom{n+\delta-1}{\delta-1}$, $\delta > 0$.

Notations

We use the following notations.

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

$$(EN)_n(t) = \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{P_k} \sum_{r=0}^{k} p_{k-r} \frac{\sin\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}}$$

INTRODUCTION

The result of Quade [9] has been generalized by several researchers like Chandra [1], Khan [5], Mohapatra & Russell [8], Sahney & Rao [10]; but most of their theorems are not satisfied for $\alpha = 1$, $p>1$. Therefore, this deficiency has provoked to investigate degree of approximation of functions belonging to Lip (α, p) considering cases $0< \alpha<1$ and $\alpha = 1$ separately. Some interesting results on Lip (α, p) class have been given by Chandra [2], Leindler [6] and Dhakal [3] by Nörlund and matrix method. Here better and sharper estimate of $f \in \text{Lip}(\alpha, p)$ than all previously known results in this direction has been determined as following:

Theorem. If $f: R \to R$ is 2π-periodic, Lebesgue integrable on $[-\pi, \pi]$ and Lip(α, p) class function for $\frac{1}{p} < \alpha \le 1,\; p \ge 1,$ then the degree of approximation of f by (E, 1)(N, p_n) means of its Fourier series (1) is given by

$$\| t_n^{EN}(x) - f(x) \|_p = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{p}}}\right), \tag{9}$$

where $t_n^{EN} = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{P_k} \sum_{r=0}^{k} p_{k-r}\, s_r$ is (E, 1)(N, p_n) means of Fourier series (1).

Lemmas

Following lemmas are needed for the proof of our theorem.

Lemma 1. If $(EN)_n(t) = \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{P_k} \sum_{r=0}^{k} p_{k-r} \frac{\sin\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}}$ then

$$(EN)_n(t) = O(n+1) \quad \text{for } 0 < t \le \frac{1}{n+1}. \tag{10}$$

Proof: $$\left|(EN)_n(t)\right| = \left| \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{P_k} \sum_{r=0}^{k} p_{k-r} \frac{\sin\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right|$$

$$\le \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{P_k} \sum_{r=0}^{k} p_{k-r} \frac{\left|\sin\left(r+\frac{1}{2}\right)t\right|}{\left|\sin\frac{t}{2}\right|}$$

$$\le \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{P_k} \sum_{r=0}^{k} (2r+1)\, p_{k-r} \qquad \because \sin\left(r+\tfrac{1}{2}\right)t \le \{2r+1\}\sin\tfrac{t}{2}$$

$$\le \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{(2k+1)}{P_k} \sum_{r=0}^{k} p_{k-r}$$

$$= \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} (2k+1) \qquad \because \sum_{k=0}^{n} p_{n-k} = P_n$$

$$= \frac{2^n (n+1)}{2^{n+1}\pi}$$

$$= O(n+1).$$

Lemma 2. If $(EN)_n(t)$ is given as in lemma 1, then

$$(EN)_n(t) = O\left(\frac{1}{t}\right) \quad \text{for } \frac{1}{n+1} < t < \pi. \tag{11}$$

Proof: $$\left|(EN)_n(t)\right| = \left| \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{P_k} \sum_{r=0}^{k} p_{k-r} \frac{\sin\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right|$$

$$\le \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{P_k} \sum_{r=0}^{k} p_{k-r} \frac{\left|\sin\left(r+\frac{1}{2}\right)t\right|}{\left|\sin\frac{t}{2}\right|}$$

$$\le \frac{1}{2^{n+1}t} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{P_k} \sum_{r=0}^{k} p_{k-r}$$

$$= \frac{1}{2^{n+1}t} \sum_{k=0}^{n} \binom{n}{k}$$

$$= O\left(\frac{1}{t}\right).$$

Proof of the Theorem

Following Titchmarsh [11], the n^{th} partial sum $s_n(x)$ of the Fourier series is given by

$$s_n(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dt\,.$$

The (N, p_n) transform of the sequence $\{s_n(x)\}$ is given by

$$t_n^N(x) - f(x) = \frac{1}{2\pi P_n}\int_0^{\pi} \phi(t)\sum_{k=0}^{n} p_k \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dt\,.$$

The (E, 1) transform of $\left\{t_n^N(x)\right\}$ is given by

$$t_n^{EN}(x) - f(x) = \int_0^{\pi}\phi(t)\frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{P_k}\sum_{r=0}^{k}p_{k-r}\frac{\sin\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,dt$$

$$= \int_0^{\pi}\phi(t)(EN)_n(t)\,dt$$

$$= \int_0^{\frac{1}{n+1}}\phi(t)(EN)_n(t)\,dt + \int_{\frac{1}{n+1}}^{\pi}\phi(t)(EN)_n(t)\,dt$$

$$= I_1 + I_2, \text{ say.} \tag{12}$$

Applying Hölder's inequality, lemma 1 and fact that $\phi(t) \in Lip(\alpha, p)$, we have

$$|I_1| \le \left[\int_0^{\frac{1}{n+1}}\left(\frac{t|\phi(t)|}{t^{\alpha}}\right)^p dt\right]^{\frac{1}{p}}\left[\int_0^{\frac{1}{n+1}}\left\{\frac{|(EN)_n(t)|}{t^{1-\alpha}}\right\}^q dt\right]^{\frac{1}{q}}$$

$$= O\left[\left(\int_0^{\frac{1}{n+1}} t^{(\alpha-1)q}\,dt\right)^{\frac{1}{q}}\right]$$

$$= O\left[\left\{\left(\frac{t^{(\alpha-1)q+1}}{(\alpha-1)q+1}\right)_{\varepsilon}^{\frac{1}{n+1}}\right\}^{\frac{1}{q}}\right], \text{ where } 0 < \varepsilon < \tfrac{1}{n}.$$

$$= O\left(\frac{1}{(n+1)^{\alpha-1+\frac{1}{q}}}\right)$$

$$= O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{p}}}\right), \tag{13}$$

where q is the conjugate index of p.

Using Hölder's inequality and lemma 2, we have

$$|I_2| \le \left[\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|}{t^{\alpha}}\right)^p dt\right]^{\frac{1}{p}} \left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{|(EN)_n(t)|}{t^{-\delta-\alpha}}\right\}^q dt\right]^{\frac{1}{q}}$$

$$= O\left[\left(\int_{\frac{1}{n+1}}^{\pi}\left(t^{-\frac{1}{p}-\delta}\right)^p dt\right)^{\frac{1}{p}}\right] O\left[\left(\int_{\frac{1}{n+1}}^{\pi} t^{(\delta+\alpha-1)q} dt\right)^{\frac{1}{q}}\right]$$

$$= O\left[\left\{\left(\frac{t^{-\delta p}}{-\delta p}\right)_{\frac{1}{n+1}}^{\pi}\right\}^{\frac{1}{p}}\right] O\left[\left\{\left(\frac{t^{(\delta+\alpha-1)q+1}}{(\delta+\alpha-1)q+1}\right)_{\frac{1}{n+1}}^{\pi}\right\}^{\frac{1}{q}}\right]$$

$$= O\left(\frac{1}{(n+1)^{-\delta}}\right) O\left(\frac{1}{(n+1)^{\delta+\alpha-1+\frac{1}{q}}}\right)$$

$$= O\left(\frac{1}{(n+1)^{\alpha-1+\frac{1}{q}}}\right)$$

$$= O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{p}}}\right) \tag{14}$$

where δ is an arbitrary number such that $q(1-\delta)-1 > 0$.

Combining the conditions (12) – (14), we have

$$\left| t_n^{EN}(x) - f(x)\right| = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{p}}}\right).$$

Now, $\left\| t_n^{EN}(x) - f(x)\right\|_p = \left[\int_0^{2\pi} \left|f(x) - t_n^{EN}(x)\right|^p dx\right]^{\frac{1}{p}}$

$$= O\left[\int_0^{2\pi}\left(\frac{1}{(n+1)^{\alpha-\frac{1}{p}}}\right)^p dx\right]^{\frac{1}{p}}$$

$$= O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{p}}}\right)\left[\int_0^{2\pi} dx\right]^{\frac{1}{p}}$$

$$= O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{p}}}\right). \qquad (15)$$

This completes proof of the theorem.

Corollary

Following corollary can be derived from the main theorem:

The degree of approximation of a function $f \in \text{Lip}\,\alpha$ by (E, 1)(N, p_n) means is given by

$$\left\|t_n^{EN}(x) - f(x)\right\|_\infty = \sup_{-\pi\le x\le\pi}\left|t_n^{EN}(x) - f(x)\right| = O\left(\frac{1}{(n+1)^\alpha}\right), \text{ for } 0<\alpha<1,$$

where, $t_n^{EN} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{P_k}\sum_{r=0}^{k} p_{k-r}\, s_r$ is the (E,1) (N, p_n) means of Fourier series (1).

REMARKS: An independent proof of corollary can be derived along the same line as the theorem.

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