FOUNDATIONS OF GEOMETRY– V

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Our senses, specially vision and touch, help us to perceive our experience of the external world. From what we observe helps us to derive logical consequences. A basic question which arises here is can we rightfully deduce about the nature of our space by observation? One can never be completely sure that a certain theory based in our experience is right. For a long time humen being on earth believed that the earth was flat. The study of geometrical facts from our observations and experience is the empirical approach to study geometry.

In analytic approach of studying geometry we represent a point by an ordered pair, triple and even by an n-tuple of real numbers. Such ordered pair, triple, and so forth are the elements of an algebraic structure known as vectors. Then different geometrical properties are equivalent to different conditions to be satisfied by specific algebraic equations. Linear algebra studies vector spaces and the results of linear algebra are translated into results concerning matrices and matrix calculations.

A revolution in mathematics began about 1900 with a new way of looking at the subject. In mathematics this new way is the axiomatic method. Certain fundamental geometrical facts, accepted without proof, were called postulates by Euclid. Also other geometrical facts to be deduced from these postulates, by means of the rules of logic, are also accepted without proof. Current mathematical usage refers to the fundamental assumptions of any mathematical theory as **axioms**. In this sense, Euclid's postulates are common notions or axioms. Several terms such as "point", "line", "circle", "right angle", and "congruence" have not been defined. Following are the Euclid's axioms (postulates):

1.   A straight line segment can be drawn joining any two points.

2.   Any straight line segment can be extended continuously in a straight line.

3.   Given any straight line segment, a circle can be drawn with the line segment as the radius and one end point as the centre.

4.   All right angles are congruent.

5.   If a straight line meets two other straight lines so that the sum of the interior angles on the same side is less than two right angles, then the two straight lines meet on that side on which the angles are less than two right angles.

Euclid's definitions are now considered inadequate. Some objections have been raised against them of which the important ones are:

(a) The fifth axiom, referred to as the parallel postulate, is much more complicated in is statement than the other axioms. In fact, this postulate can be stated in a simpler version as

(5)   Given a line   and a point P outside it, one and only one line can be drawn through P, parallel to   .

P

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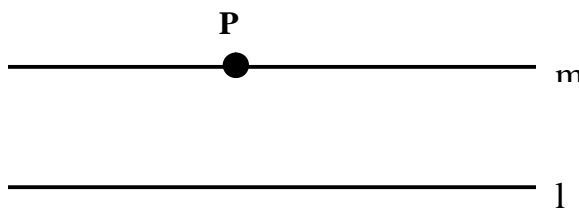
(b) The axioms of Euclid are incomplete. That is, it is not possible to prove the theorems of Euclid based entirely on the axioms of Euclid.

From mathematical point of view the first objection is not very relevant. For, there is no reason as to why complicated statements may not be taken as axioms. Nevertheless, this objection has an enormous influence on the historical development of mathematics. Attempts to remove this objection led to the development of non-Euclidean geometries. Had non-Euclidean geometry not developed the acceptance of the theory of relativity would have been difficult. Thus this objection, in fact, has contributed in the development of not only mathematics but Physics too.

The fifth postulate of Euclid looks reasonable. So reasonable, in fact, that many mathematicians wondered whether it can be proved as a theorem by using Euclid's other postulates.

One way of proving that a statement is true or not is to assume the truth of its opposite statement and then demonstrate a logical contradiction. If we try the same in case of Euclid's fifth postulate, we can assume that:

(I)   Through P no line can be drawn parallel to   .

P

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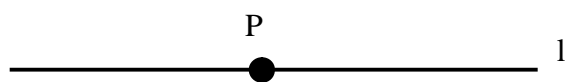
(II)   Through P more than one line can be drawn parallel to l.
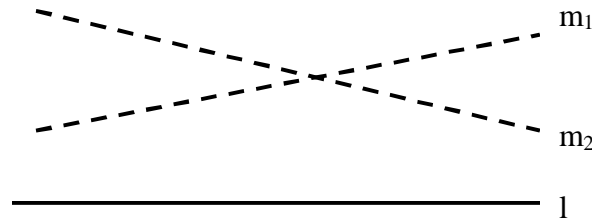
To their surprise, mathematicians found no contradictions. Geometries perfectly consistent with these assumptions were possible. These geometries, collectively known as non Euclidean geometries, were discussed and explored by Bolyai, Lobatchewsky, Gauss and Rimann in the 19th Century.

As to the second objection that Euclid's axioms are incomplete, in fact, it is not possible to prove all theorem of Euclidean geometry based on these axioms. As an example, let us try to construct an equilateral triangle with a given line segment as one side. We draw a circle with P as its centre and the given line segemt PQ as the radius.
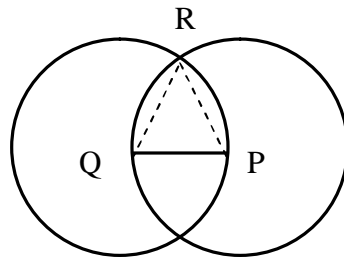
Again, draw another circle with Q as the centre and QP as the radius. Then join P and Q with a common point R on the circle to obtain the equilateral triangle PQR. But Euclid's axioms do not suggest that two circles do, indeed, have a common point.