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MATHEMATICAL MODELING OF OPTIMAL POSITION OF CHIPS ON PRINTED CIRCUIT BOARD (PCB)

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ABSTRACT

There has been a rapid development of electronic devices and competition among the electronic companies to produce smaller and lighter devices with high performance, reliability and multi-functionality. This results in the adjustment of increasing number of electronic components called chips. The demand of small shape and size of the devices needs micro analysis in this field. To adjust maximum number of chips, the space among them should be minimized. But the problem is that each of the chips can generate a significant amount of heat which can degrade the performance of the device and some of the chips could be completely damaged because of high temperature. Each of the components has its own maximum tolerate temperature and if this temperature exceeds, the component may not function properly or could be damaged completely. Thus the thermal management is an important issue for both in electronic components and the board (Printed Circuit Board) where they will be placed.

This paper presents the heat distribution on the module and based on this heat distribution a modified gradient algorithm has been used to determine the optimal position of the chips on the board. The numerical computation has been done for 1D space module and numerical results are presented graphically for 1D space module by taking specific configurations and physical properties of chips.

Key words: Printed Circuit Board (PCB), Chips, Optimal Position, Lagrange Principle, Adjoint PDE, Gradient Algorithm, Finite Differences.

INTRODUCTION

Thermal management is an important aspect in chips and printed circuit board. Heat distribution must be minimized efficiently from the small area of the board where large amount of

heat is dissipated by the multi-chips or components. So placement of chips becomes an important issue for the thermal consideration since naked chips are kept together can generate large amount of heat which can degrade the performance of the module or may cause damage of those components of the module which are highly sensitive with high temperature. The thermal performance must be accurately predicted during the design phase to get reliable and satisfactory product.

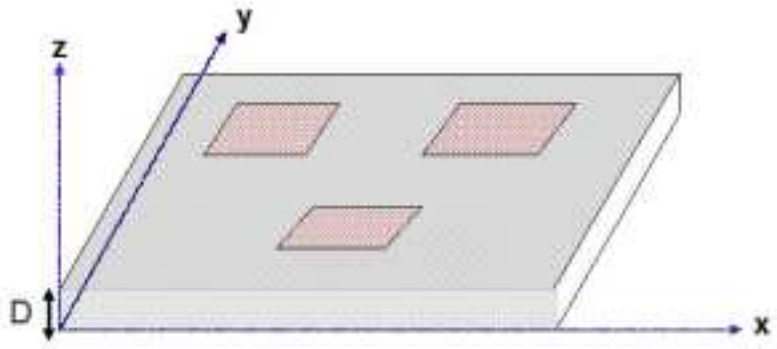

Figure 1: Schematic diagram of placement of chips in the Circuit Board.

The temperature of a chip depends on the heat it dissipates and the temperature of the region where it is located. This is also influenced by the position and physical properties of the other chips. So the position of the chips should be chosen properly. Thus, the goal of this paper is to find the position of the chips on the board in such a way that the temperature distribution among the chips is minimal.
In this paper, the derivation of the governing heat equation on the module with boundary conditions is presented. The reduction of the model from 3D-Space to 2D-Space is also shown. The cost functional of the problem which has to be minimized under PDE constraints is defined. For this, the adjoint PDE has been found using Lagrange Principle. Finally, the gradient algorithm has been used to find the optimal position of the chips to minimize the temperature distribution on the module. Solving this problem analytically or numerically in 2D space module is quite complex. Thus, the module is reduced into 1D space and problem has been solved numerically for 1D module. The numerical solution of the PDEs has been achieved by using Finite Difference Method. Using the solutions of PDEs and gradient algorithm, the optimal position of the chips on the board has been found.

MATHEMATICAL FORMULATION

Let $\Omega \subset \mathbb{R}^3$ be the bounded domain, where $\Omega := (0, L) \times (0, W) \times (0, D)$ and $L, W, D > 0$ are, respectively, the length, width and thickness of the board. In our problem, we are assuming that $L = W$. Let us take an arbitrary open subset E of Ω. Let $t \in (0, \tau), \tau > 0$, then the energy balance relation on E is given by

Heat entering through boundary faces of E + energy generation in E = storage of energy in E [5].

$$\text{The amount of heat entering through the boundary} = -\int_{\partial E} (\mathbf{q(x,t)} \cdot \mathbf{n}) ds \overset{\text{Gauss Divergence}}{=} -\int_E (\nabla \cdot \mathbf{q(x,t)}) d\mathbf{x}.$$

Where, $\mathbf{q}(\mathbf{x}, t)$ is heat flux vector on ∂E and **n** is outer unit normal vector to the boundary.

$$\text{Energy generation in E per unit time} = \int_E g(\mathbf{x}, t) d\mathbf{x}$$

Where, $g(\mathbf{x}, t)$ is the heat generation per unit volume at time t.

$$\text{Energy stored in E per unit time} = \int_E \rho C \frac{\partial T(\mathbf{x}, t)}{\partial t} d\mathbf{x}$$

Where,
$T(\mathbf{x}, t)$: the temperature on the domain Ω at time t.
C : the specific heat capacity.
ρ : the density of the material.
Substituting these expressions in energy relation, we get

$$\int_E (-\nabla \cdot \mathbf{q}(\mathbf{x}, t)) + g(\mathbf{x}, t) - \rho C \frac{\partial T(\mathbf{x}, t)}{\partial t}) d\mathbf{x} = 0 \tag{1}$$

Fourier law of heat conduction [5] states that the heat flux, the flow of heat per unit area and per unit time, at a point in a medium is directly proportional to the temperature gradient at the point, that is,

$$\mathbf{q}(\mathbf{x}, t) = -k(\mathbf{x}) \nabla T(\mathbf{x}, t) \tag{2}$$

Where $k(\mathbf{x})$ is the thermal conductivity of the material. In general it also depends on the temperature. We assume that it does not change with temperature in our module because the temperature difference in the module is small, therefore the change of conductivity is also very small and hence can be neglected.
From equations (1) and (2), we get

$$\int_E \Big(\nabla \cdot (k(\mathbf{x}) \nabla T(\mathbf{x}, t)) + g(\mathbf{x}, t) - \rho C \frac{\partial T(\mathbf{x}, t)}{\partial t}) \Big) d\mathbf{x} = 0 \tag{3}$$

Lemma 0.1 (Variational Lemma) *[4] Let $F(\boldsymbol{x})$ be a continuous function satisfying $\int_E F(\boldsymbol{x}) d\boldsymbol{x} = 0$ for every subset $E \subset \Omega \subset \mathbb{R}^n$. Then $F \equiv 0$ in Ω .*

Assuming that the integrand of equation (3) is continuous with respect to spatial variable , and applying Variational Lemma 0.1, we get

$$\nabla \cdot (k(\mathbf{x}) \nabla T(\mathbf{x}, t)) + g(\mathbf{x}, t) = \rho C \frac{\partial T(\mathbf{x}, t)}{\partial t} \qquad \mathbf{x} \in \Omega, t \in (0, \tau) \tag{4}$$

Remark 0.1 *In our setting, the integrand is not continuous on Ω because k and g are piecewise constant functions so they are in $L^1(\Omega)$. These functions can be approximated by $C^\infty(\Omega)$ because $C^\infty(\Omega) \subset L^p(\Omega), 1 \leq p < \infty$ is dense [6]. So, the integrand in equation (3) can be approximated in $C^0(\Omega)$. So, Equation (4) is true $\forall t \in (0, \tau)$ and $\forall \mathbf{x} \in \Omega$.*

Equation (4) represents the governing heat conduction equation in domain Ω with heat generation. It is second order linear partial differential equation and in particular a parabolic equation. To make the problem well posed the boundary conditions should be imposed properly.

BOUNDARY CONDITIONS

The module is exposed in the air, so the heat is convected away from it. The heat transfer rate is proportional to the overall temperature difference between the wall and the air and surface area. Let $F \subset \partial\Omega$. Then by Newton's law of cooling [5], the over all amount of the heat transfer through F due to convection is given by

$$\begin{aligned} \int_F \mathbf{q}(\mathbf{x}, t) \cdot \mathbf{n} ds &= \int_F h(T(\mathbf{x}, t) - T_a) ds \\ \Rightarrow \int_F \Big(\mathbf{q}(\mathbf{x}, t) \cdot \mathbf{n} - h(T(\mathbf{x}, t) - T_a) \Big) ds &= 0 \end{aligned} \tag{5}$$

Where, $\mathbf{n}$ is outer normal to the boundary, h is convective heat transfer coefficient and T_a is ambient temperature.
Due to Fourier Law [5],

$$\mathbf{q}(\mathbf{x}, t) = -k(\mathbf{x}) \nabla T(\mathbf{x}, t) \qquad \text{on } F \tag{6}$$

From Equations (5) and (6), we have

$$\int_F \Big(-k(\mathbf{x}) \big(\nabla T(\mathbf{x}, t) \cdot \mathbf{n} \big) - h\big(T(\mathbf{x}, t) - T_a \big) \Big) ds = 0 \tag{7}$$

Assuming that the function in the integrand of equation (7) is continuous and applying the Variational Lemma 0.1, we have

$$-k(\mathbf{x}) \big(\nabla T(\mathbf{x}, t) \cdot \mathbf{n} \big) = h(T(\mathbf{x}, t) - T_a) \qquad \mathbf{x} \in \partial\Omega, t \in (0, \tau) \tag{8}$$

Now, rewriting the governing heat equation on the module with boundary condition, we have

$$\begin{aligned} \nabla \cdot (k(\mathbf{x}) \nabla T(\mathbf{x}, t)) + g(\mathbf{x}, t) &= \rho C \frac{\partial T(\mathbf{x}, t)}{\partial t} \quad \mathbf{x} \in \Omega, t \in (0, \tau) \\ -k(\mathbf{x}) \big(\nabla T(\mathbf{x}, t) \cdot \mathbf{n} \big) &= h(T(\mathbf{x}, t) - T_a) \quad \mathbf{x} \in \partial\Omega, t \in (0, \tau) \end{aligned} \tag{9}$$

DERIVATION OF MODEL IN 2D SPACE

In the last section, we have given the mathematical formulation of the temperature distribution

in 3D case. Now, we reduce the 3D space module into 2D space which describes the average temperate distribution on 2D module. The optimal solution that is found for 2D module will approximate the optimal solution in 3D module.
The thickness of the module is relatively small compared to the length and width. So, we can assume that the physical property of the material and heat generation of the module is independent along its thickness. That means, $k = k(x, y)$ and $g = g(x, y)$.
From Remark 0.1 the function

$$F(\mathbf{x}, t) := \nabla \cdot \big(k(\mathbf{x})\nabla T(\mathbf{x}, t)\big) + g(\mathbf{x}, t) - \rho C \frac{\partial T(\mathbf{x}, t)}{\partial t} \quad \mathbf{x} \in \Omega, t \in (0, \tau)$$

is in $L^1(\Omega) \quad \forall t \in (0, \tau)$. Integrate equation (7) w.r.t z from $z = 0$ to $z = D$, we have

$$\int_{z=0}^{z=D} \Big(\nabla \cdot \big(k(x, y)\nabla T(x, y, z, t)\big) + g(x, y, t) - \rho C \frac{\partial T(x, y, z, t)}{\partial t}\Big) dz = 0$$

$$\begin{aligned} \Rightarrow \int_{z=0}^{z=D} \Big(\partial_x(k(x, y)\partial_x T(x, y, z, t)) \;&+\; \partial_y(k(x, y)\partial_y T(x, y, z, t)) + k(x, y)\partial_{zz} T(x, y, z, t) \\ &+\; g(x, y, t) - \rho C \frac{\partial T(x, y, z, t)}{\partial t}\Big) dz = 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow \partial_x(k(x, y)\partial_x \int_o^D T(x, y, z, t)dz \;&+\; \partial_y(k(x, y)\partial_y \int_o^D T(x, y, z, t)dz \\ &+\; k(x, y) \int_o^D \partial_{zz} T(x, y, z, t)dz \\ &+\; \int_o^D g(x, y, t)dz - \rho C \frac{\partial \int_o^D T(x, y, z, t)dz}{\partial t} = 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow \partial_x(k(x, y)\partial_x \widetilde{T}(x, y, t)dz \;&+\; \partial_y(k(x, y)\partial_y \widetilde{T}(x, y, t)dz \\ &+\; k(x, y)\Big(\frac{\partial T(x, y, z, t)}{\partial z}\Big|_{z=D} - \frac{\partial T(x, y, z, t)}{\partial z}\Big|_{z=0}\Big) \\ &+\; g(x, y, t) - \rho C \frac{\partial \widetilde{T}(x, y, t)}{\partial t} = 0 \end{aligned}$$

Where, $\widetilde{T}(x, y, t) := \dfrac{1}{D} \displaystyle\int_0^D T(t, x, y, z)dz$, which is in fact the average temperature along the thickness of the module. Using mean value theorem from integral calculus, there exist $\eta \in (0, D)$ such that $\dfrac{1}{D} \displaystyle\int_0^D T(t, x, y, z)dz = T(x, y, \eta, t)\frac{1}{D}\int_0^D dz = T(x, y, \eta, t)$. So,

$\widetilde{T}(x,y,t) = T(x,y,\eta,t)$.
Let us assume that the lower part of the board (that is, at $z = 0$) is insulated so that using boundary conditions at $z = 0$ and $z = D$ give

$$\frac{\partial T(x,y,z,t)}{\partial z}\Big|_{z=0} = 0 \quad \text{and} \quad \frac{\partial T(x,y,z,t)}{\partial z}\Big|_{z=D} = -\widetilde{h}(T_D - T_a) \quad \text{where,} \quad \widetilde{h} = \frac{h}{D}$$

and $T_D := T(x,y,D,t)$
Which results in the expression of the form

$$\nabla \cdot \big(k(x,y)\nabla\widetilde{T}(x,y,t)\big) - \widetilde{h}(T_D - T_a) + g(x,y,t) = \rho C\frac{\partial\widetilde{T}(x,y,t)}{\partial t} \tag{10}$$

Now, we approximate $T_D = T(x,y,D,t)$ around $z = \eta$ using Taylor's series expansion,

$$T(t,x,y,D) = T(t,x,y,\eta) + \underbrace{(D-\eta)\partial_z T(t,x,y,\eta) + \text{higher degree terms}}_{=0 \quad because \quad (D-\eta)<<1}$$

Hence, $T_D \approx \widetilde{T}$ and equation (10) becomes,

$$\nabla \cdot \big(k(\mathbf{x})\nabla\widetilde{T}(\mathbf{x},t)\big) - \widetilde{h}(\widetilde{T}(\mathbf{x},t) - T_a) + g(\mathbf{x},t) = \rho C\frac{\partial\widetilde{T}(\mathbf{x},t)}{\partial t}, \quad \mathbf{x} \in \Omega_{xy}, t \in (0,\tau)$$

Where, $\Omega_{xy} = \big\{(x,y,z) \in \Omega \,|\quad z = \eta\big\}$
Initial and boundary conditions are

$$\begin{aligned} \widetilde{T}(0,\mathbf{x}) &= T_a, \quad \mathbf{x} \in \Omega_{xy} \\ -k(\mathbf{x})\big(\nabla\widetilde{T}(\mathbf{x},t)\cdot\mathbf{n}\big) &= h\big(\widetilde{T}(\mathbf{x},t) - T_a\big), \quad \mathbf{x} \in \partial\Omega_{xy}, t \in (0,\tau) \end{aligned}$$

Remark 0.2 *The new definition for the domain $\Omega := \{(x,y) \mid (x,y,\eta) \in \Omega_{xy}\} \subset \mathbb{R}^2$, which is actually the orthogonal projection of the set Ω_{xy} on XY-plane, is used in the following sections.*

OPTIMAL CONTROL OF THE HEAT DISTRIBUTION ON THE MODULE

In this work, it has been assumed that the chips dissipate constant amount of heat for all time. The temperature distribution on the module remains stationary after some time and hence the temperature on the module remains independent of time. Thus, the optimal position of the chips on the board has been found subject to the time independent PDE constraint.
Heat distribution on the module depends on the position of the chips. So the selection of the position of the chips on the board should be chosen properly. Let $\bar{X} = (x_1, x_2, \ldots, x_m) \subset \Omega$ be the vector of mid points of the chips which are in fact the control points. We want to find the control points such that the over all heat on the module is minimal. For this, define the objective function

$$J(\widetilde{T}, \bar{X}) := \frac{1}{2} \parallel \widetilde{T} - T_a \parallel^2_{L_2(\Omega)} \quad \text{which has to be minimized} \tag{11}$$

subject to the constraints

$$\begin{aligned} -\Big(\nabla\cdot\big(k\nabla\widetilde{T}\big)-\widetilde{h}(\widetilde{T}-T_a)+g\Big) &= 0 \quad \text{in} \quad \Omega \\ k\big(\nabla\widetilde{T}\cdot\mathbf{n}\big)+h\big(\widetilde{T}-T_a\big) &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{12}$$

Since the temperature on the module depends on the position of the chips, a mapping $\bar{X} \longrightarrow \widetilde{T}(\bar{X})$ can be defined and hence the cost functional is reduced into

$$\hat{J}(\bar{X}) := \frac{1}{2} \parallel \widetilde{T}(\bar{X}) - T_a \parallel^2_{L_2(\Omega)}$$

Two more conditions have to be imposed. The chips should not leave the board (domain) and none of the chips should intersect. Mathematically these conditions are given by

(i) $\frac{l}{2} < x_{ix} < L - \frac{l}{2}$ and $\frac{l}{2} < x_{iy} < L - \frac{l}{2}$, where $x_i = (x_{ix}, x_{iy})$ and l denotes the chip-length.
(Which keeps the chips within the domain).

(ii) $\|x_i - x_j\|_\infty > l \quad \forall \quad i \neq j$
(Which does not allow chips to intersect).

We use gradient algorithm [3] to find the optimal control of our problem. To apply this algorithm we need the adjoint state variable which can be found solving the adjoint PDE of the given PDE. To find the adjoint PDE, we use Lagrange Principle [3].

LAGRANGE PRINCIPLE

The objective function

$$J(\bar{X}) = \frac{1}{2} \parallel \widetilde{T}(\bar{X}) - T_a \parallel^2_{L_2(\Omega)} \quad \text{has to be minimized}$$

subject to the constraints

$$\begin{aligned} -\Big(\nabla\cdot\big(k\nabla\widetilde{T}\big)-\widetilde{h}(\widetilde{T}-T_a)+g\Big) &= 0 \quad \text{in} \quad \Omega \\ k\big(\nabla\widetilde{T}\cdot\mathbf{n}\big)+h\big(\widetilde{T}-T_a\big) &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

We introduce two Lagrange multiplier P_1 and P_2 and Lagrange function

$$\begin{aligned} \mathscr{L}\big(\widetilde{T},\bar{X},P_1,P_2\big) &= \frac{1}{2}\int_\Omega |\widetilde{T}-T_a|^2 d\mathbf{x} - \int_\Omega -\Big(\nabla\cdot\big(k\nabla\widetilde{T}\big)-\widetilde{h}(\widetilde{T}-T_a)+g\Big)P_1 d\mathbf{x} \\ &\quad - \int_{\partial\Omega}\Big(k\big(\nabla\widetilde{T}\cdot\mathbf{n}\big)+h\big(\widetilde{T}-T_a\big)\Big)P_2 d\mathbf{s} \\ &= \frac{1}{2}\int_\Omega |\widetilde{T}-T_a|^2 d\mathbf{x} + \int_\Omega \widetilde{T}\nabla\cdot(k\nabla P_1)d\mathbf{x} + \int_{\partial\Omega} k(\nabla\widetilde{T}\cdot\mathbf{n})P_1 d\mathbf{s} \\ &\quad - \int_{\partial\Omega}\widetilde{T}k(\mathbf{n}\cdot\nabla P_1)d\mathbf{s} + \int_\Omega g P_1 d\mathbf{x} - \int_\Omega \widetilde{h}(\widetilde{T}-T_a)P_1 d\mathbf{x} \\ &\quad - \int_{\partial\Omega} k(\nabla\widetilde{T}\cdot\mathbf{n})P_2 d\mathbf{s} - \int_{\partial\Omega} h(\widetilde{T}-T_a)P_2 d\mathbf{s} \end{aligned}$$

Condition for the optimal solution is $D_{\widetilde{T}}\mathscr{L}\big(\widetilde{T}, \bar{X}, P_1, P_2\big)\xi = 0 \quad \forall\xi$
So,

$$\begin{aligned} D_{\widetilde{T}}\mathscr{L}\big(\widetilde{T}, \bar{X}, P_1, P_2\big) &= \int_\Omega (\widetilde{T} - T_a)\xi d\mathbf{x} + \int_\Omega \nabla\cdot(k\nabla P_1)\xi d\mathbf{x} + \int_{\partial\Omega} k(\nabla\xi\cdot\mathbf{n})P_1 d\mathbf{s} \\ &\quad - \int_{\partial\Omega} k(\mathbf{n}\cdot\nabla P_1)\xi d\mathbf{s} - \int_\Omega \widetilde{h}P_1\xi d\mathbf{x} - \int_{\partial\Omega} k(\nabla\xi\cdot\mathbf{n})P_2 d\mathbf{s} \\ &\quad - \int_{\partial\Omega} hP_2\xi d\mathbf{s} = 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow \int_\Omega \big(\nabla\cdot(k\nabla P_1) - \widetilde{h}P_1 + (\widetilde{T} - T_a)\big)\xi d\mathbf{x} &+ \int_{\partial\Omega} k(\nabla\xi\cdot\mathbf{n})(P_1 - P_2)d\mathbf{s} \\ &- \int_{\partial\Omega} \big(k(\nabla P_1\cdot\mathbf{n} + hP_2)\big)\xi d\mathbf{s} = 0 \end{aligned}$$

1^{st} **choice:**
$\xi \in C_0^\infty(\Omega)$, then we get

$$\begin{aligned} \int_\Omega \big(\nabla\cdot(k\nabla P_1) - \widetilde{h}P_1 + (\widetilde{T} - T_a)\big)\xi d\mathbf{x} &= 0 \\ \Rightarrow \qquad \nabla\cdot(k\nabla P_1) - \widetilde{h}P_1 + (\widetilde{T} - T_a) &= 0 \quad \text{in} \quad \Omega \end{aligned} \tag{13}$$

2^{nd} **choice:**
$\xi|_{\partial\Omega} \equiv 0$, then we get

$$\begin{aligned} \int_{\partial\Omega} k(\nabla\xi\cdot\mathbf{x})(P_1 - P_2)d\mathbf{s} &= 0 \\ \Rightarrow \qquad P_1 - P_2 &= 0 \\ \Rightarrow \qquad P_1 &= P_2 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{14}$$

Indeed, P_2 is the restriction of P_1 on the boundary $\partial\Omega$. So, we use P instead of P_1 or P_2.
3^{rd} **choice:**
$(\nabla\xi\cdot\mathbf{n})|_{\partial\Omega} = 0$ where, $\xi|_{\partial\Omega}$ is arbitrary, then we get

$$\begin{aligned} \int_{\partial\Omega} \big(k(\nabla P_1\cdot\mathbf{n}) + hP_2\big)\xi d\mathbf{s} &= 0 \\ \Rightarrow \qquad k(\nabla P_1\cdot\mathbf{n}) + hP_2 &= 0 \\ \Rightarrow \qquad -k(\nabla P_1\cdot\mathbf{n}) &= hP_2 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{15}$$

Thus, from equations (13), (14) and (15), we get the the adjoint PDE

$$\begin{aligned} \nabla(k\nabla P) - \widetilde{h}P + (\widetilde{T} - T_a) &= 0 \quad \text{in} \quad \Omega \\ -k(\nabla P\cdot\mathbf{n}) &= hP \quad \text{on} \quad \partial\Omega \end{aligned} \tag{16}$$

GRADIENT METHOD

Gradient algorithm finds the optimal solution of the cost functional (11) subject to the constrains (12). This algorithm also uses the adjoint PDE (16). In each iteration level, we have to find the search direction $\mathbf{d_n}$ and the step size α_n. In each iteration level, the algorithm finds a new position of the chips such that the cost functional decreases.
The algorithm is given by

(1) We choose the initial position of the chips, $\bar{X}_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_m^{(0)})$

(2) For k=0,1,...

(i) Solve PDE (12) for $\widetilde{T}_k$.

(ii) Solve adjoint PDE (13) for P_k.

(iii) Find the search direction $\mathbf{d_k}$.

(iv) Find the step size α_k

(v) Set $\bar{X}_{k+1} = \bar{X}_k + \alpha_k \mathbf{d_k}$

HOW TO FIND THE SEARCH DIRECTION $\mathbf{d_k}$

The search direction is given by the $\mathbf{d_k} = -\hat{J}'(\bar{X})$[3]. But $\hat{J}'(\bar{X})$ can be found from the derivative of Lagragian function with respect to $\bar{X}$, that is, $\hat{J}'(\bar{X}) = \mathscr{L}_{\bar{X}}(\widetilde{T}, \bar{X}, P)$[3]. We have,

$$\begin{aligned}
\mathscr{L}\big(\widetilde{T}, \bar{X}, P_1, P_2\big) &= \frac{1}{2}\int_\Omega |\widetilde{T} - T_a|^2 d\mathbf{x} - \int_\Omega -\Big(\nabla\cdot\big(k\nabla\widetilde{T}\big) - \widetilde{h}(\widetilde{T} - T_a) + g\Big) P d\mathbf{x} \\
&\quad - \int_{\partial\Omega} \Big(k\big(\nabla\widetilde{T}\cdot\mathbf{n}\big) + h\big(\widetilde{T} - T_a\big)\Big) P d\mathbf{s} \\
&= \frac{1}{2}\int_\Omega |\widetilde{T} - T_a|^2 d\mathbf{x} - \int_\Omega k\big(\nabla\widetilde{T}\cdot\nabla P\big) d\mathbf{x} + \int_\Omega gP d\mathbf{x} - \int_\Omega \widetilde{h}(\widetilde{T} - T_a) P d\mathbf{x} \\
&\quad - \int_{\partial\Omega} h(\widetilde{T} - T_a) P d\mathbf{x}
\end{aligned}$$

So,

$$\begin{aligned}
\mathscr{L}_{\bar{X}}(\widetilde{T}, \bar{X}, P) &= -\int_\Omega (\nabla_{\bar{X}} k)\big(\nabla\widetilde{T}\cdot\nabla P\big) d\mathbf{x} + \int_\Omega (\nabla_{\bar{X}} g) P d\mathbf{x} \\
&= -\int_\Omega (k_c - k_b)\Big(\delta\big(\mathbf{x} - (\bar{X} - \epsilon)\big) - \delta\big(\mathbf{x} - (\bar{X} + \epsilon)\big)\Big)(\nabla T\cdot\nabla P) d\mathbf{x} \\
&\quad - \int_\Omega (g_c - g_b)\Big(\delta\big(\mathbf{x} - (\bar{X} - \epsilon)\big) - \delta\big(\mathbf{x} - (\bar{X} + \epsilon)\big)\Big) P d\mathbf{x}
\end{aligned}$$

Where, $k(\mathbf{x}) = \begin{cases} k_c & \text{,if} \quad \mathbf{x} \in [\bar{X} - \epsilon, \bar{X} + \epsilon] \\ k_b & \text{,otherwise} \end{cases}$ and $g(\mathbf{x}) = \begin{cases} g_c & \text{,if} \quad \mathbf{x} \in [\bar{X} - \epsilon, \bar{X} + \epsilon] \\ g_b & \text{,otherwise} \end{cases}$

are heavy-side step functions. Where, k_c and k_b are the conductivity of the chips and board respectively, g_c and g_b are the heat generation by the chips and board respectively and $\epsilon = \frac{l}{2}$
So,

$$\begin{aligned}\mathscr{L}_{\bar{X}}(\widetilde{T}, \bar{X}, P) &= (k_b - k_c)\Big(\nabla T(\bar{X}-\epsilon)\cdot\nabla p(\bar{X}-\epsilon) - \nabla T(\bar{X}+\epsilon)\cdot\nabla p(\bar{X}+\epsilon)\Big) \\ &\quad +(g_c - g_b)\Big(P(\bar{X}-\epsilon) - P(\bar{X}+\epsilon)\Big)\end{aligned}$$

Therefore, the search direction is given by

$$\begin{aligned}\mathbf{d_k} &= (k_c - k_b)\Big(\nabla T(\bar{X}-\epsilon)\cdot\nabla p(\bar{X}-\epsilon) - \nabla T(\bar{X}+\epsilon)\cdot\nabla p(\bar{X}+\epsilon)\Big) \\ &\quad +(g_b - g_c)\Big(P(\bar{X}-\epsilon) - P(\bar{X}+\epsilon)\Big)\end{aligned}$$

HOW TO CALCULATE THE STEP SIZE α_k

The step size α_k can be found by one direction line search, which is given by $\alpha_k = min_{\alpha>0}\hat{J}(\bar{X}_k + \alpha d_k)$ [3]. But to find step size α_k by one dimension line search is somehow not so easy. So there are other ways to find the step size.

1^{st} choice:

$\alpha_k =$ constant. For example $\alpha_k = 10^{-2}$ and see the convergence and then increase or decrease the step.

2^{nd} choice:

We want to minimize the functional $\hat{J}(\bar{X}_k + \alpha d_k)$ with respect to α. For that we must have $\frac{d}{d\alpha}\hat{J}(\bar{X}_k + \alpha d_k) = 0$. Use Taylor expansion,

$$\begin{aligned}\hat{J}(\bar{X}_k + \alpha d_k) &= \hat{J}(\bar{X}_k) + \alpha\hat{J}'(\bar{X}_k)d_k + \frac{\alpha^2}{2}\hat{J}''(\bar{X}_k)[d_k, d_k] + 0 \\ \Rightarrow \hat{J}'(\bar{X}_k)d_k + \frac{\alpha}{2}\hat{J}''(\bar{X}_k)[d_k, d_k] &= \frac{\hat{J}(\bar{X}_k + \alpha d_k) - \hat{J}(\bar{X}_k)}{\alpha} \approx 0 \quad \text{for} \quad \alpha << 1 \\ \Rightarrow \alpha_k &\approx \frac{-2\hat{J}'(\bar{X}_k)d_k}{\hat{J}''(\bar{X}_k)[d_k, d_k]} \\ \Rightarrow \alpha_k &\approx \frac{2d_k^2}{\hat{J}''(\bar{X}_k)[d_k, d_k]}\end{aligned}$$

NUMERICS

The numerical solution of the problem is only done for the 1-D case, which can be considered as a cut through the board. The 1-D case gives useful solutions if the board and the chips have

the same width, i.e., each chip has only one degree of freedom. This restriction is imposed because it is not that involved and gives good results for the essential part of the problem.

Numerical Solution of the 1-D time Independent Heat Equation

In this section, we present the numerical solution of the equation (12), where we limit ourselves to 1-D case. Here, $\widetilde{T}$ is replaced by T, i.e.

$$\begin{aligned} \frac{d}{dx}\cdot(k(x)\frac{d}{dx}T(\mathrm{x}))+g(\mathrm{x})+\tilde{h}(T_a-T(x)) &= 0 && \text{in } \Omega \\ -k(x)\big(\frac{d}{dx}T(\mathrm{x})\cdot n\big) &= h(T(\mathrm{x})-T_a) && \text{on } \partial\Omega \end{aligned} \tag{17}$$

where we assume w.l.o.g. $\Omega=(0,L)$.
(Note that in 1-D the outer normal vector **n** is -1 on the left and $+1$ on the right boundary).
For this task, we use finite difference method. Therefore, we replace the differential operators by symmetric difference ones. We do this, instead of forward and backward differences, to get better results for later use in optimization. First we define an equidistant grid Ω_s of the domain Ω by the grid points $x_i=is$, $i=0,...,N$, where $s=L/N$ denotes the step size, i.e,

$$\begin{aligned} \Omega_s: &= \{x_i|x_i=is, i=1,...N-1\} \\ \partial\Omega_s: &= \{0,L\} \\ \tilde{\Omega}_s: &= \{x_i|x_i=is, i=2,...N-2\} \end{aligned}$$

This gives

$$\begin{aligned} D^0(kD^0T)-\tilde{h}T &= -(g+\tilde{h}T_a) \\ \frac{1}{4s^2}(k(x+s)(T(x+2s)-T(X))-k(x-s)(T(x)-T(x-2s))-\tilde{h}T(x) &= -(g+\tilde{h}T_a) \end{aligned}$$

This yields a system of equations

$$LT_s=f \text{ in } \tilde{\Omega}_s$$

where T_s represents the solution on the grid points. We define $m_i=-(k_{i-1}+k_{i+1}+4s^2\tilde{h})$, so the matrix L has the form

$$L=\begin{pmatrix} m_2 & 0 & k_3 & 0 & & & & \dots & & & 0 \\ 0 & m_3 & 0 & k_4 & 0 & & & \dots & & & 0 \\ k_3 & 0 & m_4 & 0 & k_5 & 0 & & \dots & & & 0 \\ & & & & & \vdots & & & & & \\ 0 & & & \dots & & 0 & k_{N-5} & 0 & m_{N-4} & 0 & k_{N-3} \\ 0 & & & \dots & & & 0 & k_{N-4} & 0 & m_{N-3} & 0 \\ 0 & & & \dots & & & & 0 & k_{N-3} & 0 & m_{N-2} \end{pmatrix}$$

The right hand side f is given by $f_i := -4s^2(g(x_i) + \tilde{h}T_a)$. Additionally, we need the boundary conditions for $\partial\Omega_s$ and via forward and backward differences the points in $\Omega_s \setminus \tilde{\Omega}_s$, since the symmetric differences would need information from outside the domain. The discretization of the boundary conditions is

$$\begin{aligned} k(x)(D^+T)(x) &= h(T(x) - T_a) \qquad \text{for the left boundary} \\ -k(x)(D^-T)(x) &= h(T(x) - T_a) \qquad \text{for the right boundary} \end{aligned}$$

Finally, we get

$$\Lambda := \begin{pmatrix} -k_0 s - hs^2 & k_0 s & & & & \\ k_1 & -(k_2 + k_1 + s^2\tilde{h}) & k_2 & & & \\ & & & L & & \\ & & & & k_{N-2} & -(k_{N-1} + k_{N-2} + s^2\tilde{h}) & k_{N-1} \\ & & & & k_N s & -k_N s - hs^2 \end{pmatrix}$$

The right hand side $\tilde{f}$ is given by

$$\tilde{f} := \begin{pmatrix} -hs^2 T_a \\ -s^2(g_1 + \tilde{h}T_a) \\ \\ f \\ \\ -s^2(g_{N-1} + \tilde{h}T_a) \\ -hs^2 T_a \end{pmatrix}$$

So the final system to be solved reads

$$\Lambda T_s = \tilde{f} \qquad \text{in } \Omega_s \cup \partial\Omega$$

CONVERGENCE OF THE METHOD

Since the solution is assumed to be regular enough i.e. here $T \in C^5$, the finite difference method used above is consistent, which can be shown by Taylor expansion! (nasty work [2])
After multiplying the system with -1, the matrix $-L$ is a L_0-matrix, since all non-diagonal elements are non-positive. Hence, the matrix $-L$ is even L-matrix, because all diagonal elements are positive. Combining this property with the strictly diagonal dominance of the matrix leads to the fact that $-L$ is an M-matrix and therefore the method is stable [2].
Note that this also guarantees the solvability of the system.
Since the method is stable and consistent, the convergence follows.

OPTIMIZATION

Numerical Solution of the 1-D Adjoint PDE

The Adjoint PDE in one dimension reads

$$\begin{aligned} \frac{d}{dx}(k\frac{d}{dx}P(x)) - \tilde{h}P + T - T_a &= 0 \quad \text{in } \Omega \\ -k(\frac{d}{dx}P(x)\cdot n) &= hP \quad \text{in } \partial\Omega \end{aligned}$$

Comparing this to the original PDE (17) and discretizing it in the same way, it can be seen that the matrix is exactly the same. The only difference is the right hand side.
Defining $f_i^{adj} := -4s^2(T(x_i) - T_a)$ it can be written as

$$\Big(0, \quad -s^2(T_1 - T_a), \quad f^{adj}, \quad -s^2(T_{N-1} - T_a), \quad 0\Big)^T$$

Therefore the numerical solution is given by

$$\Lambda P_s = \tilde{f}^{adj} \qquad \text{in } \Omega_s \cup \partial\Omega$$

Search Direction

The discretization of the search direction is again done by finite difference method via symmetric differences
$d_i = -g_i\big(P(x_i-\epsilon)-P(x_i+\epsilon)\big)-(k_B-k_i)\big(D^0T(x_i-\epsilon)D^0P(x_i-\epsilon)-D^0T(x_i+\epsilon)D^0P(x_i+\epsilon)\big)$

Step Size

We choose the step size α_k such that the distance of the movement of each chip is restricted to a certain value, i.e. $\alpha_k d_k^i \leq c \quad \forall k, i$, where i represents the index number of the chips. Therefore we define $\alpha_k := c(\max_i |d_k^i|)^{-1}$. To ensure the convergence, this value c has to be chosen small enough and α has to be replaced by an constant value for $|d_k^i|$ small enough. It is clear that this approach will not succeed for all settings. That means the constant c depends on the properties of the module.

NUMERICAL RESULTS AND DISCUSSION

It has been found that the industries use the material FR-4 (Epoxy Resin), an abbreviation for Flame Retardant 4, is a type of material used for making a printed circuit board (PCB) and the material silicon is taken for the chips. The gradient algorithm has been implemented for two different modules to locate the optimal position of the chips. In the first module, five chips have been taken of same size with same amount of heat generator. The physical properties of the FR-4 material and silicon have been presented in the table 1. In the second module, the material have been used for PCB and chips but generating different amounts of heat. The physical properties of second module are presented in the Table 2. The results for both the modules are presented graphically.

Module I

	board	chip
material	FR-4 (epoxy resin)	silicon
heat generation	$0W$	$0.4W$
conductivity	$0.002W/cmK$	$1.5W/cmK$
length	$20cm$	$2cm$
outer temp.	$293K$	
convection const.	$0.002W/cm^2K$	

Table 1: Constants

Here, we present the graphs of the temperature distribution on the module at three different levels, i.e. initial, intermediate and final iteration levels of the optimization algorithm. In this module, we have taken five chips of same size on the module with same amount of heat generator, See Table 1. The initial position of the chips is chosen arbitrarily.

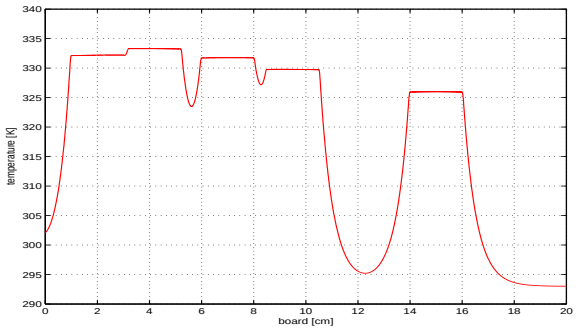

Figure 2: Temperature distribution on the module for initial position of the chips.

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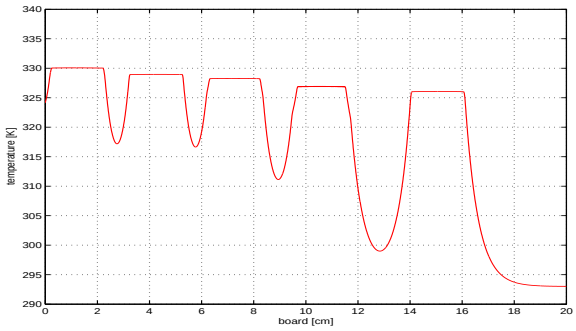

Figure 3: Temperature distribution on the module at after 7 iteration levels.

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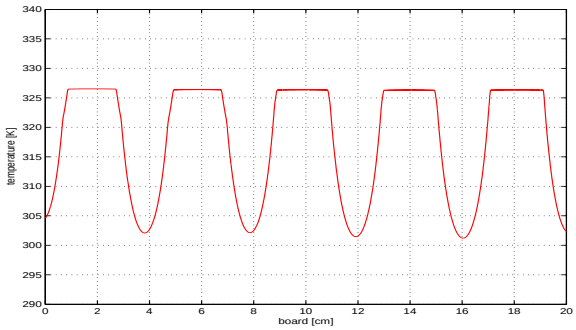

Figure 4: Temperature distribution on the module after 31 iteration levels.

Module II

In this module, we take the four chips with same physical parameter as given in module I but generating different amount of heat, that is, three chips of 0.3 W/cm and one chip of 0.6 W/cm.

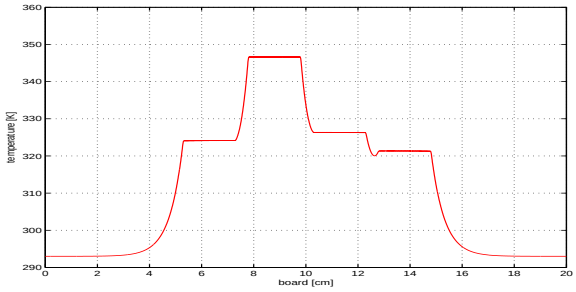

Figure 5: Temperature distribution on the module for initial position of the chips.

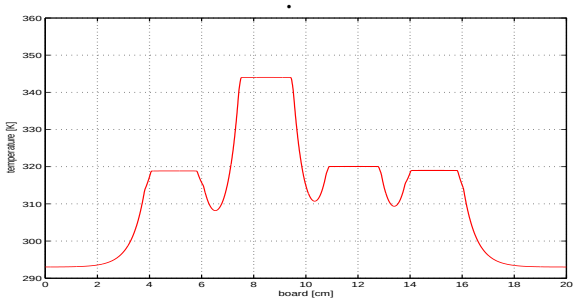

Figure 6: Temperature distribution on the module at after 7 iteration levels.

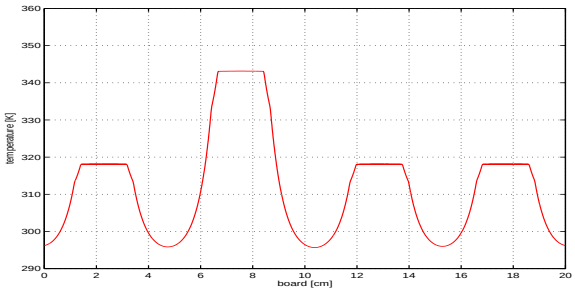

Figure 7: Temperature distribution on the module after 35 iteration levels.

From these examples, It can be concluded that the algorithm finds a better position in each iteration level. After finite number of iterations, the algorithm finds a good position of the chips on the board. From module I, it can be concluded that for the chips generating same amount of heat should be kept equidistant. In the module II, four chips are taken with different amount of heat generation and it can be seen that the chips which generate large amount of heat will be kept far away from the others by this algorithm.

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