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FOUNDATIONS OF GEOMETRY – III

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In the beginning, the geometry developed from practical problems concerning the calculations of length, areas and volumes. Instead of correct and exact mathematical methods, these computational methods were crude approximations derived from trial methods. The body of knowledge thus developed by Babylonians and Egyptians for surveying and navigation were passed on to the Greeks. The Greeks, with their scholarly pursuit, transformed this bulk of ideas into a science known as geometry. About 300 B.C. Euclid of Alexandria organized this knowledge of his time in the form of a book, the

Elements. This Elements proved such an effective book of geometry that for the next 2000 years geometers all over the world used this books as the starting point.
Euclid did not define length, distance, or inclination. He used the "rules of interaction" between the defined objects in his postulates, The five postulates of Euclid, specially the fifth one, the attraction it drew, and the various attempts to prove this postulate in terms of other postulates, has been discussed earlier.

In 1763 a man named Klugel wrote a dissertation at gottingen in which he evaluated all significant attempts made to prove the Euclid's parallel (or the fifth) postulate. Out of the 28 attempted proofs of this postulate klugel found no one to be satisfactory. Of particular interest was the work of Saccheri. He assumed the negation of the parallel postulate and hoped to derive contradiction. Out of his logical consequences he derived many strange looking results. Some of these results were inconsistent with Euclid's other postulates. People of that time did not think that there could be another geometry different from that of Euclid. Now we know that the strange looking results of Saccheri were nothing but the fundamental ideas of what we call hyperbolic geometry.

Karl F. Gauss was apparently the first mathematician who thought that the results obtained by thinking the negation of Euclid's fifth postulate will never lead to the contradiction of this postulate but these could be other geometries different than that of Euclid. Gauss' ideas seemed so revolutionary that he didn't dare publish his results due to the fear of being criticized heavily. In his words, he feared the "screams of dullards".

Not only Gauss, but Lobachevesky(1793-1856) and Bolyai (1802-1860), working independently also obtained results which were consistent but didn't lead to the contradiction of Euclid's parallel postulate. Their works were published in 1829 and 1832, respectively. But it didn't raise the screams of dullards (stupid) as Gauss expected. Actually the dullards did not pay any attention to these new theories.

Almost 40 years latter, Beltrami (1835-1900) and Felix Klein (1849-1925) were able to produce models of geometries of Lobachevsky and Bolyai. These models were produced within Euclidean geometry. Only then it was proved beyond doubt, that Euclidean geometry was free from contradiction. Not only Euclidean geometry but the geometry which was produced considering the negotiation of Euclid's fifth postulate (now called the hyperbolic geometry) was also free from any contradiction. Hyperbolic geometry also satisfied all axioms of Euclid's geometry except the parallel postulate. It was finally decided that a proof of the Euclid's parallel proof was impossible.

Now there emerged a new geometry which was different from that of Euclid's geometry. So the branching of geometry into Euclidean and non-Euclidean occurred. But there are results that are equally valid in both geometries. Also, there are results which are valid in Euclidean geometry but not in non-Euclidean geometry and vice versa. So, the theorems of geometry were divided on the basis of their dependence on parallel postulate. All theorems which do not require parallel postulate in their proofs are equally true on Euclidean geometry as well as on hyperbolic geometry. These were called the theorems of **absolute geometry**. The results of Euclidean geometry which only depend on postulates I, II, V are known as **affine geometry**. Theorems common to both absolute and affine geometries are called theorems of **ordered geometry**.

The image of a certain object can be depicted on a canvas by a painter. The image made by a painter on canvas can be regarded as a **projection** of the original object onto the canvas with the centre of projection at the eye of the painter. In the process of painting or projecting and object onto canvas length are changed or distorted depending on the relative position of the object. So there are certain geometric properties which are common in an object and in its image or projection. These geometric properties help us in recognizing an object with image in the canvas. So we want to know what these geometric properties are which do not change (invariants in mathematical term) in an object and its image. The knowledge that developed from these considerations is the **projective geometry**. The pioneer of projective geometry was a French engineer Poncelet (1788-1867). In 1813 he was a prisoner of war in Russia and was deprived of reading any books. As a result, he developed many of the basic facts of Projective geometry. In studying the properties which remain invariant in the distortion which take place in depicting and object in two dimensional form(picture or painting) we come under properties that remain invariant under the projections of parallel lines. So, affine and projective geometries are closely related. This aspect of projective geometry was recognized by Leonard Euler(1701- 1783).

Lack of computational facility was a great hindrance in the development of geometry. Descarte's (1596-1650) invention of **analytical geometry** provided simple approaches to solve many problems. As an example Theory of Conics was very complicated subject but the treatment could be made simple by analytical approach. Also this new approach help geometers to use useful and new ideas developed in Algebra and Calculus.

The scope of geometry was greatly enlarged by Riemann(1826-1866). In his great lecture of June 10, 1854, Über die Hypothesen, welche der Geometrie zu Grande Liegen (on the hypothesis which lie at the Foundations of Geometry), Georg Friedrich Bernhard Riemann developed a new view of Geometry, which we now call Riemannian geometry. He proposed a general view of geometry in which one could speak not only of two or three dimensional spaces but of any dimensional spaces. Also one could make measurements of lengths and angles. What Rieman proposed that at infinitesimal level(in a very tiny portion of space) the geometry should be exactly Eucidean. But on a very large scale these laws might vary. Riemannian geometry is the study of properties of spaces that on a microscopic level are Euclidean but on the macroscopic level may be quite different.

The general objects of study known as spaces are called Manifolds in Riemannian geometry. Manifolds are the objects of study in Modern Differential geometry. As the name suggests, the methods used to study manifolds depend on calculus. Albert Einstein used Riemannian geometry as the basis of his theory of Relativity.

How far geometric ideas are related to the physical world? Euclid believed that his geometry does reflect the true facts of the Physical world. In fact, Euclid did not believe the existence of physical entities such as lengthless, & lengthless breadth. He considered such things as intuitive properties of physical world and geometry idealized the reality.

David Hilbert (1862-1943) thought of making geometry free from physical concepts. So he rewrote the foundations of geometry. Hilbert started with undefined objects (points, lines, planes), undefined relations (collinearity, betweenness, congruence) and some axioms expressed in terms of undefined objects. Facts deduced on a logical basis from these objects and axioms are the Theorems. These Theorems do not depend on the nature of the objects but only on the axioms they satisfy.

In 1872, Felix Klein tried to classify different geometries based on the groups of transformations under which their results remain true. Since then group theory has been of great importance to geometers. Much work in this field has been done by Sophus Lie (1842-1899) and his work is known as Lie groups in his honor.

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This article is the third installment of a series article by the author