FOUNDATIONS OF GEOMETRY – II

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The ancient Greeks have already known certain defects of the Elements. Attempts to improve the treatment were made. The goal was to reduce the Euclidean postulate and axiom system to minimum.

However, all the efforts to get rid of the fifth postulate were of no avail. The typical mistake of most its proofs was either purposeful or accidental use of some or other statement not explicitly contained in the remaining postulates and axioms, and not following from them.

It is probable that Euclid tried to prove the parallel postulate. The first twenty-eight propositions of the Elements do not use the fifth postulate, as if Euclid tried to avoid using it as long as possible.

From the time of Euclid until the end of nineteenth century the problem of freeing the Euclidean theory of parallel lines from the fifth postulate was one of the most popular problems in Geometry. During this time several proofs were put forward but all of them were found erroneous.

It should be noted that despite the efforts made to prove the fifth postulate was futile, but the attempts led to many results, which helped to the establishment of logical interdependence of various geometrical statements. Several interesting and new results were found. One of such results is that **the sum of the angles of any triangle is not greater than two right angles** by A. Legendre.

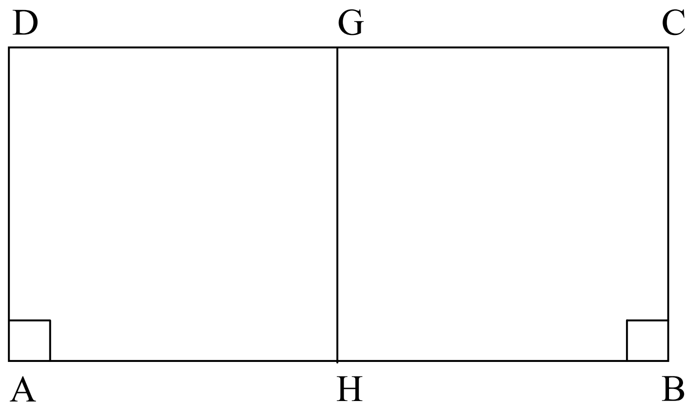
Several equivalent statements to the fifth postulate can be made. They are:

(i) All perpendiculars to one side of an acute angle cut its other side.
(ii) There exist similar triangles, which are not congruent.
(iii) There exist triangles of arbitrarily large area.
(iv) There exist triangles whose angle sum is equal to two right angles.
(v) Through a point outside a given straight line, not more than one parallel line can be drawn.

Any of these propositions can be taken as the basis of the theory of parallel lines. Assuming any one of them as well as to some other propositions to be obviously true, one can prove the fifth postulate and then, following Euclid, derive all other theorems.

Among the large number of works done by earlier geometers on the theory of parallel lines, particular mention must be made of two of them. The first one is the Italian Jesuit priest and mathematician Girolamo Saceheri (1667-1733) and the second one is the Swiss-German writer on mathematics Johann Heinrich Lambert (1728-1777). Both of them made a profound contribution to the task of substituting the theory of parallel lines.

In the book published by Saceheri entitled **Euclides ab omni naevo vindicates** (Euclid cleared of every flaw); he uses the method of **reductio ad absurdum** to prove the fifth postulate. An outline of the Saccheri's proof is as follows:



Given that a quadrilateral ABCD with two right angles at the base AD and two equal lateral sides AD and BC. HG is perpendicular to the middle point of AB.

From the symmetry of the figure about the perpendicular HG, it follows that the angles at the vertices D and C are equal. If the fifth postulate is assumed to be true, then from the Euclidean parallel theory it can be shown that the angles at D and C are right angles and hence ABCD is a rectangle.

Conversely, given in one such quadrilateral the angles at C and D are right angles, the Euclid's parallel postulate will hold. In order to prove this postulate, Saccheri proposes three possibilities that the angles C and D are either right or obtuse or acute. He gave the names to these three possibilities as right angle hypotheses, obtuse angle hypothesis and acute angle hypothesis respectively. Since the right angle hypothesis is equivalent to the parallel postulate, to prove this postulate the remaining two possibilities must be shown unacceptable. I case of obtuse angle hypothesis Saccheri came to a contradiction, and finally, in case of acute angle case he came to various conclusions which are absurd from the point of view of our understanding of geometrical ideas. For example, the acute angle hypothesis led to the conclusion that **parallel lines either possess only one common perpendicular, on both sides of which they diverge without limit, or have none, and approaching each other asymptotically in one direction, diverge in the other without limit.**

Although these propositions seem logically inadmissible because they contradict our usual concept of space, but Saccheri did not consider them illogical. After a number of precise arguments he asserted that the acute angle hypothesis was incorrect because two lines, which become closer asymptotically must have a common perpendicular at infinity. This seems contrary to the nature of a line. Also Saccheri calculated the length of a line two ways and obtained different results. Thus, he thought that the acute angle hypothesis leads to a logical contradiction. But Saccheri was not aware that he came to this conclusion due to some computational error in his calculation. As the obtuse angle as well as acute angle hypotheses was inadmissible, Saccheri concluded that the right angle hypothesis is correct and so the **proof** of the **fifth postulate** is complete.

The proof developed by Lambert in the Theory of Parallel lines in 1766 (**Die Theorie der Parallellinien**) is similar to those of Saccheri. The outline of the Lambert's proof is follows.

ABCD is a quadrilateral where the angles A, B and C are right angles. The remaining angle D may be an acute, an obtuse or a right angle. Here, again, we have three hypotheses. Having established that the right angle hypothesis is equivalent to the to the fifth postulate and the obtuse angle hypothesis leads to a contradiction, Lambert too, like Saccheri, directed his attention to the acute angle hypothesis. Lambert developed a very complicated geometrical system. The paradox concerning the location of lines in a system based on the acute angle hypothesis was similar to the one developed by Saccheri. But Lambert, unlike Saccheri, did not conclude the inadmissibility of the acute angle hypothesis simply because it contradicts the known properties of lines. Neither he committed any computational error as done by Saccheri. He did not find any logical contradiction to reject the acute angle hypothesis. Without contradicting the validity of acute angle hypothesis, the proof of the fifth postulate also hung inconclusive. So Lambert concludes that all his efforts to proof the fifth postulate came to naught. He concludes that "Proofs of the Euclidean postulate can be developed to such an extent that apparently a mere triffle remains. But careful analysis shows that "in this seeming triffle lies the crux of the matter",

In the process of developing the geometrical system concerning the acute angle hypothesis, Lambert even developed an analogy with spherical geometry and in this he saw the possibility of the acute angle hypothesis being true. He said, "I am even inclined to think that the third hypothesis is valid on some imaginary sphere. There must be some reason why it is difficult to reject it for the plane, as can be easily done with the second hypothesis."

Now let us examine the researches of the French mathematician Adrien Marie Legendre (1752-1833). Apart from his researches in geometry, Legendre is also famous for his contribution in Mechanics and Analysis.

Over a long period of time Legendre tried to prove the Euclid's fifth postulate and published several versions of his proof. Although none of the versions of the proof were found correct, Legendre's researches are of considerable interest for establishing the **connection** between the **parallel postulate** and the **sum of the interior angles of a triangle**. In Euclidean geometry the proof of the fact that the sum of the interior angles of a triangle is equal to two right angles is based on the fifth postulate.

Legendra contends that, conversely, if the sum of the interior angles of a triangle is equal to two right angles is accepted as an axiom, then the fifth postulate can be proved as a theorem. Further, Legendre considers the following three mutually exclusive possibilities in order to prove the fifth postulate with introducing any new postulates.

(1) The sum of the angles of a triangle is greater then two right angles.
(2) The sum of the interior angles of a triangle is two right angles.
(3) The sum of the interior angles of a triangle is less then two right angles.

With precise arguments, Legendre is able to show that the first possibility leads to a contradiction. He tried to show that the third possibility also leads to a contradiction. Had he been successful in doing so, then the only remaining possibility would have been the second one and which would have proved the fifth postulate. But in the process of establishing the inadmissibility of the third possibility Legendre unknowingly used the equivalent version of the fifth postulate.

Nevertheless, Legendre, in his attempt to prove the fifth postulate, established many useful results. These results are recorded here without proof.

Proposition 1: If the sum of the angles of any triangle is two right angles, the fifth postulate holds.

Proposition 2: In every triangle the sum of the interior angles of a triangle is less than or equal to two right angles. In other words, if $S(\Delta)$ denotes the sum of interior angles of a triangle, then $S(\Delta) \leq \pi$

In order to prove this proposition, the following two lemmas were proved.

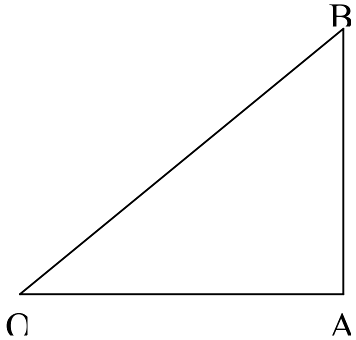
Lemma 1: In every triangle the sum of two interior angles is less then two right angles.

Lemma 2: For any given triangle and a preassigned angle of this triangle another triangle can be constructed such that the sum of the angles of both triangles are the same but one angle of the constructed triangle is half of the preassigned angle of the given triangle.

Proposition 3: If the sum of the interior angles of at least one triangle is equal to two right angles, then so is the sum of the interior angles of any other triangle.

Proposition 3 states that given two triangles ABC and A’B’C’ where the sum of the interior angles of triangle ABC is equal to two right angles, then so is the sum of the interior angles of triangle A’B’C’. If we can prove that there exists at least one triangle whose sum of the interior angles is equal to two right angles, then by Proposition 3 in every triangle the sum of the interior angles would be equal to two right angles, and then, by proposition 1, the **fifth postulate follows**.

Here we cite one example. Of there is an acute angle such that the perpendicular drawn at any point on one of its arms intersects the other arm, then the sum of the interior angles if this triangle is equal to two right angles.

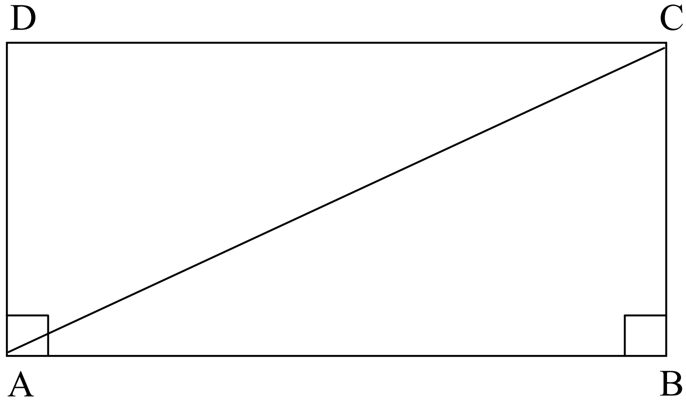


Take any arbitrary angle whose vertex is at O. draw a perpendicular BA at a point A on the arm OA and let this perpendicular meets the arm OB at B. Then it can be proved that the sum of the interior angles of Δ OAB is equal to two right angles.

This is an alternative version of the fifth postulate. So do we take this as a proof of the parallel postulate?

The week point in this argument is the justification that the perpendicular at A meets OB at B. It had been shown, by careful analysis, that the existence of the point B cannot be established without using the fifth postulate.

We would like to draw the attention of the readers to the close connection between the arguments of Legendre and those of Saccheri and Lambert. The sum of the angles of a triangle is great than two right angles, is equal to two right angles, is less then two right angles correspond to the obtuse angle, right angle and acute angle hypotheses of Saccheri and Lambert. Let us consider the Saccheri's quadrilateral where the angles A and B are both right angles.



It is already proved that the angles at C and D are equal. The there possibilities that the angles C and D are either obtuse angles, or right angles or acute angles exist. By joining the diagonal AC we can divide the figures into two triangles DAC and BAC where the sum of the three angles in Δ DAC are either greater than or equal to or less than two right angles according to the three possibilities which Saccheri considered.

Conversely, if we assume that the sum of three angles of a triangle is greater than or equal to or less than two right angles, then these possibilities correspond with the obtuse angle or the right angle or the acute angle hypothesis of Saccheri or Lambert.

Finally, it is easy to see that Saccheri's right angle hypothesis and Legendre's assumption that the sum of the three angles of a triangle are both equivalent to the fifth postulate.

Legendre tried hard to show that there exists no triangle whose sum of the angles is less than two right angles. This is equivalent to Saccheri's failure to find the contradiction for the acute angle case.

Proposition 1-3 outlined here are attributed to Legendre by tradition. But it does not mean that Saccheri and Lambert did not know it. Actually, Saccheri and Lambert went considerably farther than Legendre. But Legendre formulated them in more clear terms.